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AN IMPROVED ESTIMATOR FOR SCALE PARAMETER OF CLASSICAL PARETO DISTRIBUTION

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ABSTRACT

An improved estimator for scale parameter of classical pareto distribution utilizing the information of shape parameter has been proposed. The additional information may be obtained either from the investigator's past experience or from similar type of survey done by some reliable agencies. The expressions for bias and mean square error of the proposed estimator have been derived and comparison with the usual unbiased has been made.

Introduction :

Many socioeconomic and other naturally occurring quantities are distributed according to certain statistical distribution with very long right tail. Pareto distribution fits very well on most of such distribution. In particular for the distribution of income over a population, pareto distribution is an appropriate model.

The probability density function of classical pareto distribution is

$$f(x;a,\sigma)=a\sigma^{a}x^{-(a+1)} \qquad x\geq\sigma, a>0 \tag{1}$$

where a and σ are shape and scale parameter respectively. Let X_1, X_2, ..., X_n be a random sample of size n from a classical pareto population whose probability density function is given by (1).

For unknown a the unbiased estimator of a is given by

$$\hat{\sigma}_u=\left[1-\frac{1}{(n-1)\hat{a}}\right]\hat{\sigma}$$

where, $\hat{\sigma} = X_{(1)} = \min(X_1, X_2, \ldots. X_n)$

and

$$\hat{a} = \left[\frac{1}{n}\sum_{i=1}^{n}\log\frac{X_i}{X_{(i)}}\right]^{-1}$$

are MLE of σ and a respectively.

If $a = a_0$ is known, an unbiased estimator of σ is $\tilde{\sigma}_u = \left[1 - \frac{1}{na_0}\right]\hat{\sigma}$.

The problem of estimation of scale parameter of classical pareto distribution have been considered by different authors like Likes [1], Saxena & Johnson [2], Rohtagi and Saleh [3] and others when a prior information of parameter is not available. Sometimes the research worker may have some information about the parameter of the population prior to sample data for investigation. This information may be obtained either from the investigator's past experience or from similar type of surveys done by some reliable agencies.

The use of this additional information results in increase in the precision of the estimator. Recently Singh & others [4] considered the problem of estimation for scale parameter of a classical pareto distribution when its prior information is known. In present paper we have studied the property of the estimator of the scale parameter using prior information of the shape parameter.

2. The Proposed Estimator :

Suppose a guess value a_0 of the shape parameter a is available. A preliminary test may be conducted from hypothesis H0 : $a = a_0$ against $H_1 : a \neq a_0$ for estimating scale parameter σ. If α is pressigned level of significance, the hypothesis H_0 is accepted if

$$r_1 \leq \frac{2na_0}{\hat{a}} \leq r_2$$

Where r_1 and r_2 are such that

$$P\left[\chi^2_{2(n-1)} > r_2\right] = \frac{\alpha}{2}$$

$$P\left[\chi^2_{2(n-1)} < r_1\right] = \frac{\alpha}{2}$$

and $\chi^2_{2(n-1)}$ is a chi-square variate with 2(n–1) degrees of freedom.

Therefore, the proposed estimator of σ is

$$\hat{\sigma}_{PT} = \begin{cases} k[1-\dfrac{1}{(n-1)\hat{a}}]\hat{\sigma} + (1-k)[1-\dfrac{1}{na_0}]\hat{\sigma} \ ; & \text{if } t_1 \le a \le t_2 \\ [1-\dfrac{1}{(n-1)\hat{a}}]\hat{a} & ;\text{otherwise} \end{cases}$$

where $t_1 = \dfrac{2a_0 n}{r_2}$ and $t_2 = \dfrac{2a_0 n}{r_1}$ and k is a constant, known as Shrinkage factor.

3. Bias, Mean Square Error and Relative Efficiency of $\hat{\sigma}_{PT}$

3.1 Bias : Since $\hat{a}$ and $\hat{\sigma}$ are independently distributed the value of $E[\hat{\sigma}_{PT}]$ is written as

$$E[\hat{\sigma}_{PT}] = \int_{t_1}^{t_2}\int_{\sigma}^{\infty}[k\{1-\frac{1}{(n-1)\hat{a}}\}\hat{\sigma} + (1-k)\{1-\frac{1}{na_0}\}\hat{\sigma}]f_1(\hat{a})f_2(\hat{\sigma})d\hat{a}d\hat{\sigma}$$

$$+\int_{0}^{t_2}\int_{\sigma}^{\infty}[1-\frac{1}{(n-1)\hat{a}}]\hat{\sigma}f_1(\hat{a})f_2(\hat{\sigma})d\hat{a}d\hat{\sigma}$$

$$+\int_{t_2}^{\infty}\int_{\sigma}^{\infty}[1-\frac{1}{(n-1)\hat{a}}]\hat{\sigma}f_1(\hat{a})f_2(\hat{\sigma})d\hat{a}d\hat{\sigma}$$

where,

$$f_1(\hat{a}) = \frac{(na)^{n-1}}{\Gamma(n-1)(\hat{a})^n} e^{-\frac{na}{\hat{a}}} \qquad \hat{a} > 0$$

and

$$f_2(\hat{\sigma}) = \frac{na\sigma^{na}}{(\hat{\sigma})^{na+1}} \ , \qquad \hat{\sigma} > \sigma$$

are the probability density functions of $\hat{a}$ and $\hat{\sigma}$ respectively.

On evaluating and simplifying, we have

$$E[\hat{\sigma}_{PT}] = \sigma + (1-k)\frac{a\sigma}{na-1}\left[\frac{Q}{a} - \frac{P}{a_0}\right] \tag{2}$$

Where $$P = \frac{1}{\Gamma(n-1)}\left[I_{q_2}(n-1) - I_{q_1}(n-1)\right]$$

$$Q = \frac{1}{\Gamma n}\left[I_{q_2}(n) - I_{q_1}(n)\right]$$

and $$q_1 = \frac{na}{t_2} = \frac{ar_1}{2a_0}, \quad q_2 = \frac{na}{t_1} = \frac{ar_2}{2a_0}$$

$$I_q(m) = \int_0^q e^{-t} t^{m-1} dt$$

Therefore,

Bias $= E[\hat{\sigma}_{PT}] - \sigma$

$$= (1-k)\frac{a\sigma}{na-1}\left[\frac{P}{a} - \frac{Q}{a_0}\right]$$

3.2 Mean Square Error (MSE) :

Mean Square Error of $\hat{\sigma}_{PT}$ is given by

$$MSE[\hat{\sigma}_{PT}] = E[\hat{\sigma}_{PT}^2] - 2\sigma E[\hat{\sigma}_{PT}] + \sigma^2 \tag{3}$$

Now,

$$E[\hat{\sigma}_{PT}] = \int_{t_1}^{t_2}\int_{\sigma}^{\infty}[k^2\{1 - \frac{1}{(n-1)\hat{a}}\}\hat{\sigma}^2 + (1-k)^2(1 - \frac{1}{na_0})^2\hat{a}^2$$

$$+ 2k(1-k)\{1 - \frac{1}{(n-1)\hat{a}}\}(1 - \frac{1}{na_0})\hat{\sigma}^2]f_1(\hat{a})f_2(\hat{\sigma})d\hat{a}d\hat{\sigma}$$

$$+\int_{0}^{t_1}\int_{\sigma}^{\infty}[1-\frac{1}{(n-1)\hat{a}}]^2\hat{\sigma}^2 f_1(\hat{a})f_2(\hat{\sigma})d\hat{a}d\hat{\sigma}$$

$$+\int_{t_2}^{\infty}\int_{\sigma}^{\infty}[1-\frac{1}{(n-1)\hat{a}}]^2\hat{\sigma}^2 f_1(\hat{a})f_2(\hat{\sigma})d\hat{a}d\hat{\sigma}$$

On evaluating and simplifying we obtain

$$E\left[\hat{\sigma}_{PT}^2\right]=[1+\frac{1}{(n-1)(na-2)a}]\sigma^2$$
$$+[\{k^2+(1-k)^2(1-\frac{1}{na_0})^2+2k(1-k)(1-\frac{1}{na_0})-1\}P$$
$$-2\{k^2+k(1-k)(1-\frac{1}{na_0})-1\}\frac{1}{na}Q$$
$$+(k^2-1)\left[1-\frac{1}{n(n-1)a^2}R\right]\frac{na}{na-2}\sigma^2 \tag{4}$$

where,

$$R=\frac{1}{\Gamma(n+1)}\left[I_{q_2}(n+1)-I_{q_1}(n+1)\right]$$

Using (2) and (4) in (3) and simplifying, we have

$$MSE[\hat{\sigma}_{PT}]=\sigma^2[1+\frac{1}{(n-1)(na-2)a}]\sigma^2$$
$$+[\{k^2+(1-k)^2(1-\frac{1}{na_0})^2+2k(1-k)(1-\frac{1}{na_0})-1\}P$$
$$-2\{k^2+k(1-k)(1-\frac{1}{na_0})-1\}\frac{1}{na}Q$$

$$+(k^2-1)\frac{1}{n(n-1)a^2}R]\frac{na}{na-2}\sigma^2$$

$$-2\sigma^2[1+\frac{a}{na-1}(1-k)\{\frac{Q}{a}-\frac{P}{a_0}\}]$$

Also, $\text{MSE}[\hat{\sigma}_u]=\frac{\sigma^2}{(n-1)(na-2)a}$; $na>2$

4. The Relative Efficiency :

The relative efficiency of $\hat{\sigma}_{PT}$ with respect to $\hat{\sigma}_u$ is defined as

$$\text{REF}[\hat{\sigma}_{PT},\hat{\sigma}_u]=\frac{\text{MSE}[\hat{\sigma}_u]}{\text{MSE}[\hat{\sigma}_{PT}]}$$

Thus

$$\text{REF}\ [\hat{\sigma}_{PT}.\hat{\sigma}_u]=$$

$$\frac{1}{(n-1)(na-2)a}\left[\frac{1}{(n-1)(na-2)a}+\frac{na}{(na-2)}[\{k^2+(1-k)^2(1-\frac{1}{na_0})^2\right.$$

$$+2k(1-k)(1-\frac{1}{na_0})-1\}P-2\{k^2+(1-k)(1-\frac{1}{na_0})-1\}\frac{Q}{na}$$

$$\left.+(k^2-1)\frac{R}{n(n-1)a^2}]-\frac{2a}{na-1}(1-k)\{\frac{Q}{a}=\frac{P}{a_0}\}\right]^{-1}$$

The values of relative efficiency for different values of k, a, a_0, n and α are presented in table 1 to 3. These tables show that the preliminary test estimator (Improved Estimator) $\hat{\sigma}_{PT}$ has higher efficiency than $\hat{\sigma}_u$ when the guess value a_0 is near to a and n, k are small and gain in efficiency decreases with increase of α for these set of values a, a_0, n and k.

Values of REF $[\hat{\sigma}_{PT}, \hat{\sigma}_u]$

Table 1

n = 5, α = 0.001

k	a=1.25	a = 0.75	a = 0.90	a = 1.00	a = 1.00	a = 2.5
	a_0 = 0.50	a_0 = 1.00	a_0 = 1.05	a_0 = 1.00	a_0 = 0.80	a_0 = 1.25
0.1	0.4755	0.9937	1.1027	1.2033	1.2464	0.6574
0.2	0.4291	1.0077	1.1060	1.1961	1.2372	0.6743
0.3	0.4048	1.0186	1.1062	1.1841	1.2221	0.6944
0.4	0.3964	1.0263	1.1019	1.1677	1.2016	0.7183
0.5	0.4020	1.0306	1.0935	1.1471	1.1762	0.7467
0.6	0.4228	1.0313	1.0819	1.1230	1.1465	0.7805
0.7	0.4640	1.0285	1.0654	1.0956	1.1134	0.8208
0.8	0.5383	1.0223	1.0464	1.0657	1.0775	0.8692
0.9	0.6791	1.0127	1.0244	1.0336	1.0394	0.9279

Table 2

n = 5, α = 0.05

k	a=1.25	a = 0.75	a = 0.90	a = 1.00	a = 1.00	a = 2.5
	a_0 = 0.50	a_0 = 1.00	a_0 = 1.05	a_0 = 1.00	a_0 = 0.80	a_0 = 1.25
0.1	0.1464	0.9642	1.0220	1.0079	0.7787	0.2334
0.2	0.0674	0.9746	1.0177	1.0106	0.7839	0.1894
0.3	0.0484	0.9836	1.0216	1.0124	0.7928	0.1685
0.4	0.411	0.9911	1.0238	1.0133	0.8057	0.1599
0.5	0.0387	0.9969	1.0242	1.0133	0.8228	0.1606
0.6	0.0397	1.0010	1.0228	1.0124	0.8448	0.1706
0.7	0.0446	1.0034	1.0197	1.0106	0.8723	0.1939
0.8	0.0574	1.0040	1.0148	1.0080	0.9063	0.2432
0.9	0.0971	1.0029	1.0082	1.0044	0.9483	0.3668

Table 3

n = 7, α = 0.001

k	a=1.25	a = 0.75	a = 0.90	a = 1.00	a = 1.00	a = 2.5
	a_0 = 0.50	a_0 = 1.00	a_0 = 1.05	a_0 = 1.00	a_0 = 0.80	a_0 = 1.25
0.1	0.2504	0.9795	1.0452	1.0618	0.7926	0.1008
0.2	0.0168	0.9880	1.0499	1.0612	0.7962	0.0738
0.3	0.0099	0.9980	1.0521	1.0589	0.8036	0.624
0.4	0.0078	1.0057	1.0518	1.0550	0.8151	0.0576
0.5	0.0070	1.0110	1.0490	1.0495	0.8308	0.0571
0.6	0.0071	1.0138	1.0438	1.0425	0.8514	0.0605
0.7	0.0078	1.0141	1.0361	1.0339	0.8755	0.0696
0.8	0.0101	1.0119	1.0262	1.0239	0.9100	0.0903
0.9	0.0175	1.0072	1.0141	1.0126	0.9502	0.1512

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