# Electrostatic Problem Related to Two Equal infinite circular Strips: Charged to an unknown constant potential 

Dr. Suman Kumar Verma<br>Department of Mathematics, School of Open Learning<br>University of Delhi, Delhi-110007<br>INDIA


#### Abstract

The solution of the electrostatic problem involving two equal co-axialsperfectly conducting infinite circular strips placed in a semi-infinite space bounded by a grounded plane, charged to equal or equal and opposite potential is discussed here. Using Green's Function approach, its solution is first reduced to that of a Fredholm integral equation of first kind which is readily solved by making a simple transformation of variables. Further, the expressions for the unknown potential and charge density are derived.


## INTRODUCTION

Two-dimensional electrostatic problems involving two parallel straight infinite strips have been discussed by many authors using different techniques. Tranter [1] discussed the electrostatic problem involving two equal, parallel, coplanar infinite strips charged to potentials +1 in a free space. Further, when these strips are placed symmetrically inside a long grounded circular cylindrical shell of a large radius or outside a grounded circular cylinder of small radius, Srivastava and Gupta [2] studied the perturbation in the charge density of the two strips. They used finite Hilbert transform techniques given by Srivastava and Lowengrub[3]. Goel and Jain [4, 5] presented the solutions of four twodimensional electrostatic problems involving two equal, parallel, coplanar perfectly conducting infinite strips charged to potentials +1 or an unknown constant potential when the strips are placed symmetrically inside or outside a long grounded circular cylinder using perturbation technique of Jain and Kanwal[6]. Goel and Jain [7] also solve the same problem in the semi-infinite space bounded by a grounded infinite plane perpendicular to the two parallel, non-coplanar staggered strips using the same technique. Also, Lal and Jain [8] Presented the solution of the problem of tow equal, parallel, non-coplanar staggered conducting infinite strips charge to prescribed constant potentials in the space bounded by two grounded infinite planes which are parallel to the two strips. The same procedure is adopted, to solve the problems containing two equal, parallel coplanar strips, charged to equal or equal and opposite constant potentials, when these are placed symmetrically outside a grounded

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elliptic cylinder in its principal plane which contains the major axis and four non-intersecting conducting strips places symmetrically inside a long grounded circular cylinders or in a free space, by Lal and Jain. [9, 10].

Now, I present the solution of the electrostatic problem involving two equal co-axials perfectly conducting infinite circular strips placed in a semi-infinite space bounded by a grounded plane, charge to equal or equal, and opposite potentials. Following the usual Green's function approach, the solution of each of these two problems is first reduced to that of a Fredholm integral a simple transformation of variables when the involved perturbation parameter is small. Further, the expressions for the unknow potential and the charge density are derived in each case, when the total charge per unite height on each strip is +1 or -1 . Solutions of electrostatic problems also seem to be new.

## Electrostatic Problem: Two equal co-axial circular strip under Constant potential

Consider the two-dimensional electrostatic problem of two equal co-axial charged infinite circular strips of radius a, symmetrically placed in a semi-infinite space bounded by a grounded plane $\mathrm{X}=-\mathrm{C}(\mathrm{C}>\mathrm{A})$. The strips are supposed to be perfectly conducting and are charged to an unknown constant potential A so that the total charge per unit height on each strip is unity. In cylindrical polar co-ordinates ( $\mathrm{r}, 0, \mathrm{z}$ ) the two strips are defined by $\mathrm{r}=\mathrm{a}, \beta<|\theta|<\alpha, 0<\beta<\alpha<\pi,-\infty<\mathrm{z}<\infty$, (see Fig. 1), where a is the radius of the circular strips.

Thus, we must solve the following two-dimensional boundary value problem for the electrostatic potential $\phi(\mathrm{r}, \theta)$
$\nabla^{2} \phi(\mathrm{r}, \theta)=0$, in D,
$\phi(\mathrm{a}, \theta)=\mathrm{a}, \beta<|\theta|<\alpha, 0<\beta<\alpha<\pi$,
$\phi(-\mathrm{c} \sec \theta, \theta)=0, \pi / 2<|\theta| \leq \pi$
$\phi, \frac{\partial \phi}{\partial \mathrm{r}}$ are continuous across the two circular $\operatorname{arcs} \mathrm{r}=\mathrm{a}$,

$$
0 \leq|\theta|<\beta, \alpha<|\theta| \leq \pi,
$$

where D is the whole region of the $(\mathrm{r}, \theta)$ plane lying.
outside the arcs $\mathrm{r}=\mathrm{a}, \beta<|\theta|<\alpha$ and is bounded by the line $\mathrm{x}=-\mathrm{c}$ and $\mathrm{A}(\mathrm{A}>0)$ is the unknown constant potential of the two strips. the integral representation formula for $\phi(\mathrm{r}, \theta)$ follows from the usual Green function approach. The function $\phi$ satisfying (1), and is given by

$$
\begin{equation*}
\phi(r, \theta)=a \int_{-\alpha}^{-\beta}+\int_{\beta}^{\alpha} \sigma\left(a, \theta_{1}\right) \mathrm{G}\left(a, \theta \mid a, \theta_{1}\right) d \theta_{1}, \tag{5}
\end{equation*}
$$

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where $\sigma\left(a, \pm \theta_{1}\right), \beta<\theta_{1}<\alpha$, are the unknown charge densities of the two strips and Green's function $G$, as given by Stakgold [11] is

$$
\begin{align*}
\mathrm{G}\left(r, \theta \mid r_{1}, \theta_{1}\right) & =-\frac{1}{4 \pi} \log \left(r^{2}+r_{1}^{2}-2 r r_{1} \cos \left(\theta-\theta_{1}\right)\right) \\
& +\frac{1}{4 \pi} \log \left(r^{2}+r_{1}^{2}-2 r r_{1} \cos \left(\theta+\theta_{1}\right)\right) \\
& +4 c^{2}+4 c\left(r \cos \theta+r_{1} \cos \theta_{1}\right) \tag{6}
\end{align*}
$$

Also, from the symmetry of the problem, we have

$$
\begin{equation*}
\sigma\left(a, \theta_{1}\right)=\sigma\left(a,-\theta_{1}\right)=\sigma\left(\theta_{1}\right), \beta \theta_{1}<\alpha \tag{7}
\end{equation*}
$$

Finally, on using boundary condition (2) in (5) and the relation (7), we obtain the following governing Fredholm integral equation of the first kind for this problem.

$$
\begin{equation*}
a \int_{\beta}^{\mathrm{ca}}\left[\sigma\left(a, \theta_{1}\right) G\left(a, \theta \mid a, \theta_{1}\right)+\sigma\left(a,-\theta_{1}\right) G\left(a, \theta|a,-\theta| a,-\theta_{1}\right)\right] d \theta=\mathrm{A}, \beta<\theta<a . \tag{8}
\end{equation*}
$$

When we substitute in this integral equation the value of Green's function G given by the relation (6), we obtain after some simplifications.

$$
\begin{array}{r}
\left.a \int_{\beta}^{\alpha} \sigma\left(\theta_{1}\right)\left[\left.\log \frac{a^{2}}{4 c^{2}}+\log \right\rvert\, \cos \theta-\cos \theta_{1}\right) \right\rvert\,-\lambda\left(\cos \theta+\cos \theta_{1}\right) \\
\left.-\frac{\lambda^{2}}{4}\left(1-2 \cos ^{2} \theta-2 \cos \theta \cos \theta_{1}-\cos 2 \theta_{1}\right)+0\left(\lambda^{3}\right)\right] d \theta_{1}=-2 \pi \mathrm{~A} \tag{9}
\end{array}
$$

Where the dimensionless parameter $\lambda=\mathrm{a} / \mathrm{c}$ is assumed to be small. The form of the above integral equation suggests that the unknown density function $\sigma\left(\theta_{1}\right)$ and the unknown constant A can be expressed in the form.

$$
\begin{align*}
& \sigma\left(\theta_{1}\right)=\sigma_{0}\left(\theta_{1}\right)+\lambda \sigma_{1}\left(\theta_{1}\right)+\lambda^{2} \sigma_{2}(\theta)+0\left(\lambda^{3}\right)  \tag{10}\\
& \mathrm{A}=\mathrm{A}_{0}+\lambda \mathrm{A}_{1+} \lambda^{2} \mathrm{~A}_{2}+0\left(\lambda^{3}\right) \tag{11}
\end{align*}
$$

Equating the coefficients of equal powers of $\lambda$ on both sides of (9), we get.

$$
\begin{align*}
& \left.a \int_{\beta}^{\alpha} \sigma_{0}\left(\theta_{1}\right)\left[\log \frac{a^{2}}{4 c^{2}}+\log \left|2\left(\cos \theta-\cos \theta_{1}\right)\right|\right) d \theta_{1}\right) \\
& =-2 \pi \mathrm{~A}_{0} / a, \beta<\theta<\alpha,  \tag{12}\\
& a \int_{\beta}^{\alpha} \sigma_{1}\left(\theta_{1}\right)\left[\log -\frac{a^{2}}{4 c^{2}}+\log \left|2\left(\cos \theta-\cos \theta_{1}\right)\right|\right) \\
& =-2 \pi \mathrm{~A}_{1} / a+\int_{\beta}^{\alpha} \sigma_{0}\left(\theta_{1}\right)\left(\cos \theta+\cos \theta_{1}\right) d \theta_{1} \beta<\theta<\alpha,  \tag{13}\\
& \left.a \int_{\beta}^{\alpha} \sigma_{2}\left(\theta_{1}\right)\left[\log \frac{a^{2}}{4 c^{2}}+\log \left|2\left(\cos \theta-\cos \theta_{1}\right)\right|\right) d \theta_{1}\right)
\end{align*}
$$

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$$
\begin{align*}
& =-2 \pi \mathrm{~A}_{2} / a+\int_{\beta}^{\alpha} \sigma_{1}\left(\theta_{1}\right)\left(\cos \theta+\cos \theta_{1}\right) d \theta_{1}, \beta<\theta<\alpha, \\
& =\frac{1}{4} \int_{\beta}^{\alpha} \sigma_{0}\left(\theta_{1}\right)\left(\cos 2 \theta+2 \cos \theta \cos \theta_{1}+\cos 2 \theta_{1}\right) d \theta_{1}, \beta<\theta<\alpha, \tag{14}
\end{align*}
$$

and so on.
All the above equations (12) to (14) are of the same type and therefore can be solved by integral equation techniques explained has various authors[12,13,14]have solved Fredholm integral equation.

$$
\begin{equation*}
\sigma_{0}\left(\theta_{1}\right)=-\frac{2 A_{0}}{\alpha}\left(\frac{\sin \theta_{1}}{\Delta \mathrm{R}\left(\theta_{1}\right)}\right), \beta<\theta_{1}<\alpha \tag{15}
\end{equation*}
$$

where

$$
\begin{gather*}
\Delta=\log \left(a^{2}(\cos \beta-\cos \alpha) / 8 c^{2}\right),  \tag{16}\\
\mathrm{R}\left(\theta_{1}\right)=\left(\left(\cos \beta-\cos \theta_{1}\right)\left(\cos \theta_{1}-\cos \alpha\right)\right)^{1 / 2} \tag{17}
\end{gather*}
$$

and we have used the substitutions
$\cos \theta_{1}=[(\cos \beta-\cos \alpha) \cos y+(\cos \beta+\cos \alpha)] / 2$,
$\cos \theta=[(\cos \beta-\cos \alpha) \cos x+(\cos \beta+\cos \alpha)] / 2$
when we substitute the value of $\sigma_{0}\left(\theta_{1}\right)$ from (15) in the right-hand side of (13), we get.

$$
\begin{align*}
& \left.\quad \int_{\beta}^{\alpha} \sigma_{1}\left(\theta_{1}\right)\left[\log \frac{a^{2}}{4 c^{2}}+\log \left|2\left(\cos \theta-\cos \theta_{1}\right)\right|\right) d \theta_{1}\right) \\
& =-2 \pi \mathrm{~A}_{1} / a-\frac{A_{0} \pi}{a \Delta}(\cos \beta+\cos \alpha-2 \cos \theta), \beta<\theta<\alpha, \tag{19}
\end{align*}
$$

Inverting equation (19), we get

$$
\begin{align*}
& \sigma_{0}\left(\theta_{1}\right)=-\frac{\sin \theta_{1}}{\alpha \Delta \mathrm{R}\left(\theta_{1}\right)}\left[2 \mathrm{~A}_{1}+2 \mathrm{~A}_{0}\left\{\frac{1}{2}\left(1+\frac{2}{\Delta}\right)(\cos \beta+\cos \alpha)-\cos \theta_{1}\right\}\right], \\
& \beta<\theta_{1}<\alpha \tag{20}
\end{align*}
$$

Again, on substituting the values of $\sigma_{0}\left(\theta_{1}\right)$ and $\sigma_{1}\left(\theta_{1}\right)$ from relations (15) and (2) in the right-hand side of (14), we get.

$$
\begin{align*}
& \left.\int_{\beta}^{\alpha} \sigma_{2}\left(\theta_{1}\right) \frac{a^{2}}{4 c^{2}}+\log \left|2\left(\cos \theta-\cos \theta_{1}\right)\right|\right) \mathrm{d} \theta_{1}, \\
& \quad=\frac{\pi}{a \Delta}(\mathrm{~L}+\mathrm{M} \cos \theta+\mathrm{N} \cos 2 \theta)_{,} \beta<\theta_{1}<\alpha \tag{21}
\end{align*}
$$

where

$$
\begin{aligned}
& \mathrm{L}=-2 \Delta \mathrm{~A}_{2}-\left(\frac{\cos \beta+\cos \alpha}{2}\right)\left(2 A_{1}+A_{0}(\cos \beta+\cos \alpha)\left(1+\frac{2}{\Delta}\right)\right) \\
& +\frac{\mathrm{A}_{0}}{4}\left(3 \cos ^{2} \alpha+3 \cos ^{2} \beta+2 \cos \beta \cos \alpha\right),
\end{aligned}
$$

$$
\begin{align*}
& \mathrm{M}=-2 \mathrm{~A}_{1}-\mathrm{A}_{0}(\cos \beta+\cos \alpha)\left(\frac{2}{\Delta}+\frac{1}{2}\right), \\
& \mathrm{N}=\mathrm{A}_{0} / 2 . \tag{22}
\end{align*}
$$

Inverting equation (21), we obtain

$$
\begin{aligned}
& \sigma_{2}\left(\theta_{1}\right)=\frac{1}{a \Delta} \frac{\sin \theta}{\mathrm{R}\left(\theta_{1}\right)}\left[\frac{1}{\Delta}(\mathrm{~L}+\mathrm{P})+\mathrm{M}\left\{-\cos \theta_{1}+\frac{1}{2}(\cos \beta+\cos \alpha)\right\}\right. \\
& +\mathrm{N}\left\{2 \cos \theta_{1}(\cos \beta+\cos \alpha)-4 \cos ^{2} \theta_{1}+\left(\cos ^{2} \beta+\cos ^{2} \alpha\right) / 2\right.
\end{aligned}
$$

$$
\begin{equation*}
-\cos \beta \cos \alpha\}], \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{P}=\mathrm{M}(\cos \beta+\cos \alpha) / 2+\mathrm{N}^{2} /\left(2 \mathrm{~A}_{\mathrm{O}}\right) \tag{24}
\end{equation*}
$$

Relations (10, (15), (20) and (23) give the required value of the charge density $\sigma\left(\theta_{1}\right)$ in terms of the unknown constants $\mathrm{A}_{\mathrm{O}}, \mathrm{A}_{1}$ and $\mathrm{A}_{2}$ which are evaluated by substituting this value of $\sigma\left(\theta_{1}\right)$ in the following relation which gives the total charge per unit height on each strip to be unity:

$$
\begin{equation*}
a \int_{\beta}^{\alpha} \sigma\left(\theta_{1}\right) d \theta_{1}=1 \tag{25}
\end{equation*}
$$

Thus, we have

$$
\begin{equation*}
\mathrm{A}=-\frac{1}{2 \pi}\left[\Delta-(\cos \beta+\cos \alpha) \lambda-\frac{p \pi}{\Delta} \lambda^{2}+0\left(\lambda^{3}\right)\right]+\cdots \tag{26}
\end{equation*}
$$

Expression for the capacity of the condenser formed by the two circular strips and the bounding grounded plane $x=-c$, is given by the formula.

$$
\begin{equation*}
\mathrm{C}=\frac{2}{\mathrm{~A}}, \tag{27}
\end{equation*}
$$

where A is given by (26)
When $\beta \rightarrow 0$, in the above result (27), we obtain the expression for the capacity of the condenser formed by a circular strip $\mathrm{r}=\mathrm{a},-\alpha<\theta<\alpha, 0<\alpha<\pi,-\infty<\mathrm{z}<\infty$ and the bounding grounded plane $x=-c$ as.

$$
\begin{equation*}
C_{1}={ }_{\beta \rightarrow 0}^{L t} \frac{2}{A}=-\frac{1}{2 \pi}\left[\Delta_{1}-(1+\cos \alpha) \lambda-\frac{P_{1} \pi}{\Delta_{1}} \lambda^{2}+0\left(\lambda^{3}\right)\right] \tag{28}
\end{equation*}
$$

where

$$
\begin{aligned}
& \Delta_{1}=2 \log \left(\frac{\alpha \sin \frac{\alpha}{z}}{2 v}\right), \\
& \mathrm{P}_{1}=\frac{\Delta_{1}}{16 \pi}\left[2(1+\cos \alpha)^{2}-1\right] .
\end{aligned}
$$

Also, when we let $\mathrm{c} \rightarrow \infty$ l. $\varepsilon ., \lambda \rightarrow 0$, in the result (10), we obtain the solution of the corresponding, problem of two equal co-axial perfectly conducting infinite circular strips when these are charged to constant potential A in a free space:

$$
\sigma^{\prime}\left(\theta_{1}\right)=\left[\sigma\left(\theta_{1}\right)\right]_{\lambda \rightarrow 0}=\frac{\sin \theta_{1}}{a \pi\left[\left(\cos \beta-\cos \theta_{1}\right)\left(\cos \theta_{1}-\cos \alpha\right)\right]^{1 / 2}}
$$

$$
\begin{equation*}
\beta<\theta_{1}<\alpha . \tag{29}
\end{equation*}
$$



Fig. 1 Geometry of the Problem

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