DYNAMIC STIFFNESS FORMULATION AND FREE VIBRATION ANALYSIS OF AN ORTHOTROPIC PLATE BY CLASSICAL PLATE THEORY

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ABSTRACT

Free vibration analysis of orthotropic plate has been investigated with the help of dynamic stiffness method using classical plate theory. The rectangular plates have two opposite edges simply-supported, while all possible combinations of free, simply-supported and clamped boundary conditions are applied to the other two edges. Hamilton's principle is used to derive the governing differential equations of motion and natural boundary conditions in free vibration. The dynamic stiffness matrix is derived by relating the amplitudes of forces to those of the displacements at the plate ends. The Wittrick–Williams algorithm is used as the solution technique when applying the dynamic stiffness matrix to compute the natural frequencies and mode shapes.

Keywords: Dynamic Stiffness Matrix, FEM, Free Vibration, Penalty Method, Wittrick-Williams Algorithm.

I INTRODUCTION

Aircraft structures are generally modelled as assemblies of thin-walled structural elements. In particular, the top and bottom skins, torsion box, ribs and webs of the wing are idealised as plates. Thus the free vibration analysis of such structures plays an important role in aircraft design. The analysis facilitates aeroelastic and response analyses. The purpose of this paper is to develop the dynamic stiffness method for an accurate and efficient free vibration analysis of an orthotropic plates and plate assemblies.

FEM is an approximate method, but it generally converges to the exact solution with increasing number of elements. However, the accuracy of results cannot be always guaranteed. This is particularly true in dynamic analysis at high frequencies when the FEM may become unreliable. Thus, there is, and there will always be a need to use analytical methods based on classical theories, wherever possible, to validate the FEM which provide further insights and importantly, restore confidence in design. One such method is that of the dynamic stiffness method and (DSM) which gives exact results that are independent of the number of elements used in the analysis. For instance, one single structural element can be used in the DSM to compute any number of natural frequencies to any desired accuracy, which of course, is impossible in the FEM. In DSM, once initial assumptions about the displacement field have been made, no inaccuracy occurs in the analysis.

A strong point about DSM is that it has all the essential features of FEM such as coordinate transformation, offset connections, assembly procedure, etc., and yet it retains the exactness of results through the use of exact solution of the governing differential equation. However, the solution techniques for FEM and DSM are different. Unlike the conventional FEM which leads to a linear eigen value problem, the DSM leads to a non-linear eigen value problem which is generally solved by applying the Wittrick–Williams algorithm.

The current investigation is carried out in following steps. First, the fundamental equation of the CPT for orthotropic plate is briefly summarized. Secondly, the dynamic stiffness matrix based on the CPT is formulated. Subsequent to this development, the assembly procedure and imposition of boundary conditions by suppressing appropriate degrees of freedom (penalty method) are explained in Section. This is followed by Section which highlights the application of the Wittrick–Williams (WW) algorithm for computation of natural frequencies of thick plates with various boundary conditions. Once the DS matrices using CPT has been derived, the results are computed for rectangular plates with two opposite sides simply supported and the others having any generic boundary conditions (BC) which can be in any combination of clamped (C), free (F), or simply supported (SS). Finally the paper closes with some concluding remarks.

II MATHEMATICAL FORMULATION

2.1. Geometrical configuration

The paper deals with orthotropic rectangular plates defined by a thickness h, length L and width b. The plate has two opposite edges simply-supported along y axis (i.e. along the edges $x = 0$ and $x = L$), while the other two edges may be free, simply-supported, or clamped. The position of any point of the mid surface of the plate is described by two Cartesian coordinates x and y. The origin of the coordinate system is located at the center of the plate. The orthotropic axes of the material are parallel with axes x and y.

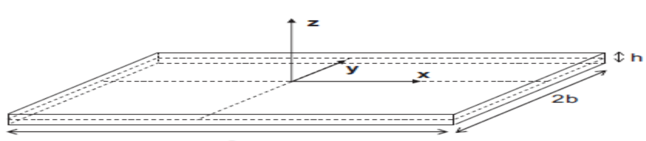

Fig. 1.Rectangular plate

2.2. Kirchhoff-Love assumptions

1. Cross sections of the plate perpendicular to the middle plane prior to deformation remain plane and perpendicular to the deformed middle plane after the deformation shown in the fig.1 below.

2. This implies that in-plane displacements are linear functions of curvature and the thickness coordinate.

3. The thickness of the plate being assumed constant.

4. The normal strain in the direction perpendicular to the middle plane is equal to zero.

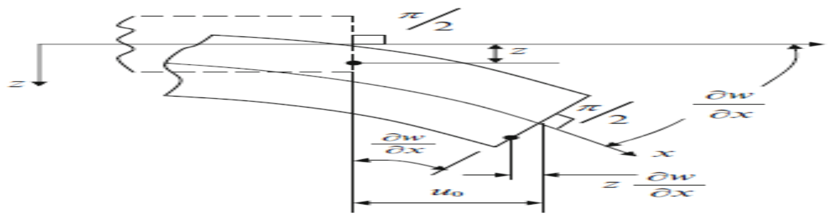

Fig. 2.Deformations of plate in the xz-plane

Following the Kirchhoff-Love assumption, displacements of an arbitrary point of the plate are

$$\mathbf{u} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u^0(x,y) - z\dfrac{\partial w^0}{\partial x} \\ v^0(x,y) - z\dfrac{\partial w^0}{\partial y} \\ w^0(x,y) \end{Bmatrix} \qquad (1)$$

where$\frac{\partial w^0}{\partial x} = \emptyset_y$ and$\frac{\partial w^0}{\partial y} = \emptyset_x$are the rotational displacements about the *x* and *y* axes at the middle surface of the plate, respectively,w^0 isthe transverse displacement and $u^0$and $v^0$are displacements of the middle plane or the displacements in the membrane mode.

The main focus of this work is on the out of plane vibratory motion of the plate so that the displacements in the membrane modeu_o(x, y) and v_o(x, y) are excluded. Clearly the only unknown in the above expression is the vertical displacement$w^0(x,y)$.

The strains in the middle plane can be obtained as linear functions of the displacements as:

$$\begin{Bmatrix} \varepsilon_{11} = \varepsilon_{xx} = \dfrac{\partial u}{\partial x} = -zk_x = -z\dfrac{\partial^2 w^0}{\partial x^2} \\ \varepsilon_{22} = \varepsilon_{yy} = \dfrac{\partial v}{\partial x} = -zk_y = -z\dfrac{\partial^2 w^0}{\partial y^2} \\ \varepsilon_{12} = \varepsilon_{xy} = \dfrac{\gamma_{xy}}{2} = \dfrac{1}{2}\left[\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right] = -zk_{xy} = -z\dfrac{\partial^2 w^0}{\partial x \partial y} \end{Bmatrix} \qquad (2)$$

For orthotropic plate we have the constitutive relations which are obtained by considering the plane stress assumption. In the orthotropic axes (1, 2, and 3), these relations are given by the following equations:

$$\sigma \equiv \begin{Bmatrix} \sigma_{11} = \sigma_{xx} \\ \sigma_{22} = \sigma_{yy} \\ \sigma_{12} = \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} (Q_{11}\epsilon_{11} + Q_{12}\epsilon_{22}) \\ (Q_{12}\epsilon_{11} + Q_{22}\epsilon_{22}) \\ (Q_{66}\epsilon_{12}) \end{Bmatrix} \qquad (3)$$

whereσ_{11} ,σ_{22} σ_{12} are thecomponents of the Cauchy stress tensor and ϵ_{11} ,ϵ_{22} $\epsilon_{12}$are the components of the small strain tensor. Material constantsQ_{ij} are given by the following equations:

$$\{Q_{ij}\} \equiv \left\{\begin{matrix} Q_{11} = \dfrac{E_1}{(1-\vartheta_{12}\vartheta_{21})} \\ Q_{22} = \dfrac{E_2}{(1-\vartheta_{12}\vartheta_{21})} \\ Q_{12} = \dfrac{\vartheta_{12}E_2}{(1-\vartheta_{12}\vartheta_{21})} \\ Q_{66} = G_{12} \end{matrix}\right\} \qquad (4)$$

whereE_1, E_2 are Young's moduli along the orthotropicdirections1and2, respectively, ϑ_{12},ϑ_{21} are major and minor Poisson's ratios and G_{12} is the shear modulus.

The 3D plate problem is essentially reduced to a 2D problem by integrating the stresses along the thickness of the plate. The forces associated with out-of-plane displacements are as follows

2.3 Stress Resultants

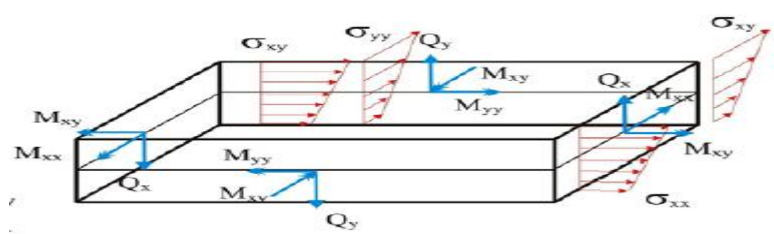

Fig. 3.Stress and Moment Resultants

Stress resultants due to normal stresses:

$$\{N\} \equiv \{N_i\} = \begin{Bmatrix} N_{xx}(x,y) \\ N_{yy}(x,y) \\ N_{xy}(x,y) \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx}\ (x,y,z) \\ \sigma_{yy}\ (x,y,z) \\ \sigma_{xy}\ (x,y,z) \end{Bmatrix} dz \qquad (5)$$

Stress resultants due to transverse shear stresses:

$$\{Q\} \equiv \{Q_i\} = \begin{Bmatrix} Q_{xx}(x,y) \\ Q_{yy}(x,y) \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz}\ (x,y,z) \\ \sigma_{yz}\ (x,y,z) \end{Bmatrix} dz \qquad (6)$$

Moment resultants:

$$\{M\} \equiv \{M_i\} = \begin{Bmatrix} M_{xx}(x,y) \\ M_{yy}(x,y) \\ M_{xy}(x,y) \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx}\ (x,y,z) \\ \sigma_{yy}\ (x,y,z) \\ \sigma_{xy}\ (x,y,z) \end{Bmatrix} zdz \qquad (7)$$

The small strains, obtained from the displacement field Eq. (2), are introduced in the constitutive equations Eq. (3) to obtain stress/displacement relationships. Then, internal force displacement relationships are obtained from internal force definitions. In the case of plates, for which the orientation of the constitutive material is such that orthotropic axes 1 and 2are equal to axes x and y, respectively, we obtain the following equations:

$$M_{xx} = -\frac{h^3}{12}Q_{11}\frac{\partial^2 w^0}{\partial x^2} - \frac{h^3}{12}Q_{12}\frac{\partial^2 w^0}{\partial y^2}, \qquad M_{yy} = -\frac{h^3}{12}Q_{12}\frac{\partial^2 w^0}{\partial x^2} - \frac{h^3}{12}Q_{22}\frac{\partial^2 w^0}{\partial y^2}$$

$$M_{xy} = -2\frac{h^3}{12}Q_{66}\frac{\partial^2 w^0}{\partial x \partial y} \qquad (8)$$

2.4. Equations of motion

By using the analysis of an Infinitesimal Plate Element the well-known equations of motion for plates is given as follows:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = \rho h \frac{\partial^2 w^0}{\partial t^2} \qquad (9)$$

Where ρ is the mass density of the material

By introducing the constitutive Eq. (3) in Eq. (4), the following governing differential equation of an orthotropic plate in free vibration is obtained as:

$$D_x \frac{\partial^4 w^0}{\partial x^4} + D_{xy}\frac{\partial^4 w^0}{\partial x^2 \partial y^2} + \frac{\partial^4 w^0}{\partial y^4} = \rho h \frac{\partial^2 w^0}{\partial t^2} \qquad (10)$$

Where D_x, D_y and D_{xy} are flexural rigidities and which are given as:

$$D_x = -\frac{h^3}{12}Q_{11}, D_{xy} = -\frac{h^3}{6}Q_{12} - 4\frac{h^3}{12}Q_{66}, D_y = -\frac{h^3}{12}Q_{22} \qquad (11)$$

2.5. Boundary conditions

Free edge boundary conditions are considered to solve the formulation of the dynamic problem. These boundary conditions describe external transverse forces and bending moments along the edges of the plate. This Boundary Conditions (BC) is given as:

$$V_x = D_x \frac{\partial^3 w^0}{\partial x^3} + \left(\frac{D_{xy}}{2} - \frac{h^3}{6}Q_{66}\right)\frac{\partial^3 w^0}{\partial x \partial y^2}$$

$$M_x = D_x \frac{\partial^2 w^0}{\partial x^2} - \frac{h^3}{12}Q_{12}\frac{\partial^2 w^0}{\partial y^2} \qquad (12)$$

III DYNAMIC STIFFNESS DEVELOPMENT

The first step in developing the DS matrix is to solve the governing differential Eq. (6). The solution is sought in the traditional Levi form. A levy type solution which satisfies the BCs is sought in the following form:

$$w_0(x, y, t) = \sum_{m=1}^{\infty} W_m(x) e^{i\omega t} \sin(\alpha_m y) \qquad (13)$$

Whereω is the unknown frequency and$\alpha_m = \frac{m\pi}{L}$ (m=1, 2 …∞)

By substituting Eq. (8) into Eq. (5), the following fourth order ordinary differential equations are obtained:

$$D_x \frac{d^4 W_m}{dx^4} - \alpha_m^2 D_{xy} \frac{d^2 W_m}{dx^2} + (\alpha_m^4 D_y + \rho h \omega^2) W_m = 0 \qquad (14)$$

The solution of the above equation is obtained by using the differential equation method which gives the four roots. From these roots two solutions of the differential equation are possible:

Case 1.if $\alpha_m^2 \geq \sqrt{\alpha_m^4 (D_{xy}^2 - 4D_x D_y) - 4D_x \rho h \omega^2}$ →all roots are real $(r_{1_m}, -r_{1_m}, r_{2_m}, -r_{2_m})$

$$r_{1_m} = \sqrt{\frac{\alpha_m^2 D_{xy} + \sqrt{\alpha_m^4 (D_{xy}^2 - 4D_x D_y) - 4D_x \rho h \omega^2}}{2D_x}}$$

$$r_{2_m} = \sqrt{\frac{\alpha_m^2 D_{xy} - \sqrt{\alpha_m^4 (D_{xy}^2 - 4D_x D_y) - 4D_x \rho h \omega^2}}{2D_x}}$$

The solution is:

$$W_m(x) A_m \cosh(r_{1_m} x)\, B_m sinh(r_{1_m} x) C_m \cosh(r_{1_m} x)\, D_m sinh(r_{1_m} x) \qquad (15)$$

Case 2.if $\alpha_m^2 \leq \sqrt{\alpha_m^4 (D_{xy}^2 - 4D_x D_y) - 4D_x \rho h \omega^2}$ →

Two real and two imaginary roots $(r_{1_m}, -r_{1_m}, ir_{2_m}, -ir_{2_m})$

$$r_{1_m} = \sqrt{\frac{\alpha_m^2 D_{xy} + \sqrt{\alpha_m^4 (D_{xy}^2 - 4D_x D_y) - 4D_x \rho h \omega^2}}{2D_x}}, r_{2_m} = \sqrt{\frac{-\alpha_m^2 D_{xy} + \sqrt{\alpha_m^4 (D_{xy}^2 - 4D_x D_y) - 4D_x \rho h \omega^2}}{2D_x}}$$

The solution is:

$$W_m(x) = A_m \cosh(r_{1_m} x) + B_m sinh(r_{1_m} x) + C_m \cosh(r_{2_m} x) + D_m sinh(r_{2_m} x) \qquad (16)$$

The procedure to obtain the DS matrix for the first case is given below. Same procedure is applied to obtain the DS matrix for second case.

Now from the known displacement w^0 (Eq. (15) and (13)), the rotation $\emptyset_y$, the edge reactions or net shear force V_x, and bending moment M_{xx} can be expressed in the following form using Eq. (12).

Rotation:

$$\emptyset_{y_m}(x, y) = \emptyset_{y_m}(x) \sin(\alpha_m y) = \frac{-\partial W_m(x)}{\partial x} \sin(\alpha_m y)$$

$$\emptyset_{y_m}(x, y) = -[\, A_m r_{1_m} \sinh(r_{1_m} x) + B_m r_{1_m} cosh(r_{1_m} x) C_m r_{2_m} sinh(r_{2_m} x) D_m r_{2_m} cosh(r_{2_m} x)\,] \sin(\alpha_m y) \qquad (17)$$

Net shear force:

$$V_{x_m}(x, y) = v_{x_m}(x) \sin(\alpha_m y) = \left[D_x \frac{\partial^3 w^0}{\partial x^3} + \left(\frac{D_{xy}}{2} - \frac{h^3}{6} Q_{66} \right) \frac{\partial^3 w^0}{\partial x \partial y^2} \right] \sin(\alpha_m y)$$

$$\text{Let,} H = \left(\frac{D_{xy}}{2} - \frac{h^3}{6} Q_{66}\right)$$

$$\begin{aligned} V_{x_m}(x,y) = {} & \left[A_m \sinh(r_{1_m} x)\left(D_x r_{1_m}^3 - H r_{1_m} \alpha_m^2\right)\right. \\ & + B_m \cosh(r_{1_m} x)\left(D_x r_{1_m}^3 - H r_{1_m} \alpha_m^2\right) \\ & + C_m \sinh(r_{2_m} x)\left(D_x r_{2_m}^3 - H r_{2_m} \alpha_m^2\right) \\ & \left. + D_m \cosh(r_{2_m} x)\left(D_x r_{2_m}^3 - H r_{2_m} \alpha_m^2\right)\right] \sin(\alpha_m y) \end{aligned} \tag{18}$$

Bending moment:

$$\begin{aligned} M_{xx_m}(x,y) &= M_{xx_m}(x) \sin(\alpha_m y) \\ &= \left(D_x \frac{\partial^2 w^0}{\partial x^2} - \frac{h^3}{12} Q_{12} \frac{\partial^2 w^0}{\partial y^2}\right) \sin(\alpha_m y) \end{aligned}$$

Let, $I = -\frac{h^3}{12} Q_{12}$

$$\begin{aligned} M_{xx_m}(x,y) = {} & \left[A_m \cosh(r_{1_m} x)\left(D_x r_{1_m}^2 + \alpha_m^2\right) + B_m \sinh(r_{1_m} x)\left(D_x r_{1_m}^2 + \alpha_m^2\right)\right. \\ & + C_m \cosh(r_{2_m} x)\left(D_x r_{2_m}^2 + \alpha_m^2\right) \\ & \left. + D_m \sinh(r_{2_m} x)\left(D_x r_{2_m}^2 + \alpha_m^2\right)\right] \sin(\alpha_m y) \end{aligned} \tag{20}$$

The boundary conditions for displacements are:

$$\text{at } x = 0 \ \rightarrow \ W_m = W_1 \ ; \quad \emptyset_{y_m} = \emptyset_{y_1} \,,$$

$$\text{at } x = b \ \rightarrow \ W_m = W_2 \ ; \quad \emptyset_{y_m} = \emptyset_{y_2} \,, \tag{21}$$

similarly the BC for the forces is:

$$\text{at } x = 0 \ \rightarrow \ v_{x_m} = -V_1 \ ; \quad M_{xx_m} = -M_1 \,,$$

$$\text{at } x = b \ \rightarrow \ v_{x_m} = V_2 \ ; \quad M_{xx_m} = M_2 \,, \tag{22}$$

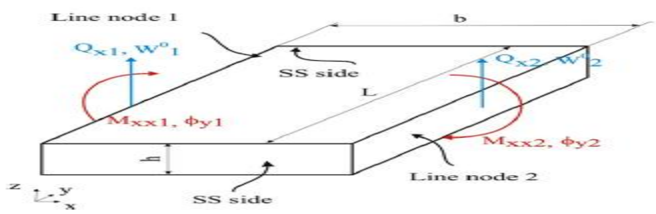

Fig. 4.Boundary conditions for displacements and forces for a plate element.

By applying these BCs for displacements, i.e. substituting Eq. (21) into Eqs. (15) and (17), we get the following equations:

This can be written in the matrix relationship as:

$$\begin{bmatrix} W_1 \\ \emptyset_{y_1} \\ W_2 \\ \emptyset_{y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -r_{1_m} & 0 & -r_{2_m} \\ Ch_1 & Sh_1 & Ch_2 & Sh_2 \\ -r_{1_m} Sh_1 & -r_{1_m} Ch_1 & -r_{2_m} Sh_2 & -r_{2_m} Ch_2 \end{bmatrix} \begin{Bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{Bmatrix} \tag{23}$$

Where, $Ch_1 = \cosh(r_{1_m} b)$, $Ch_2 = \cosh(r_{2_m} b)$

$Sh_1 = sinh(r_{1_m} b)\,, Sh_2 = \sinh(r_{2_m} b)$

$[\delta] = [A]\{C\}$ (24)

Subsequently, by applying the BCs for forces, i.e. substituting Eq. (22) into Eqs. (18) and (19), the following matrix relationship is obtained:

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & R_1 & 0 & R_2 \\ L_1 & 0 & L_2 & 0 \\ -R_1 S_{h_1} & -R_1 C_{h_1} & -R_2 S_{h_2} & -R_2 C_{h_2} \\ -L_1 C_{h_1} & -L_1 S_{h_1} & -L_2 C_{h_2} & -L_2 C_{h_2} \end{bmatrix} \begin{Bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{Bmatrix} \quad (25)$$

$[F] = [R]\{C\}$ (26)

Where, $R_i = D_x r_{i_m}^3 - H\, r_{i_m} \alpha_m^2$, $L_i = -D_x r_{i_m}^2 - I\, \alpha_m^2$ $with\ i = 1, 2.$

Using Eqs. (24) and (26) the dynamic stiffness matrix K for the plate element based on the CPT can be obtained by eliminating the constant vector C to give:

$[F] = [K]\{\delta\}$ (27)

Where, $[K] = [R]\,[A]^{-1}$ (28)

Thus we get 4×4dynamic stiffness matrix from Eq. (28) with six independent terms as$s_{vv}\,, s_{vm}\,, s_{mm}\,, f_{vv}\,, f_{vm}\,, f_{mm}$ which describe the effect on shear and moment due to unit displacements. Thus **K** can be expressed as:

$$[K] = \begin{bmatrix} s_{vv} & s_{vm} & f_{vv} & f_{vm} \\ s_{vm} & s_{mm} & -s_{vm} & f_{mm} \\ f_{vv} & -f_{vm} & s_{vv} & -s_{vm} \\ f_{vm} & f_{mm} & -s_{vm} & s_{mm} \end{bmatrix} \quad (29)$$

Explicit expressions of the elements are derived by extensive algebraic manipulation using Matlab. These are given in the Appendix.

IV APPLICATION OF DSM

4.1. Assembly procedure and application of boundary conditions

The dynamic stiffness matrix given by Eqs... (29) is the key point to compute exact natural frequencies of levy type of plates which are simply supported on two opposite sides. We can determine the natural frequencies and mode shapes by using classical method for individual plates but, we cannot apply this method for plate assemblies. This can be done in the DSM approach. This is similar to the finite element method which is schematically shown in Fig. 5.Here each element of plate is connected through nodal lines instead of single points. From this overall master stiffness matrix will be banded as in the case of FEM.

Boundary conditions can be applied in the same way as we apply in the finite element method. The penalty method is generally used to apply the boundary conditions to suppress the particular degree of freedom. In this method, a large value of stiffness is added to the appropriate term on the leading diagonal of the dynamic stiffness matrix. The procedure for applying the boundary conditions is summarized as follows:

- Free (F): no penalty is applied.
- Simply supported (SS): W_i is penalised.
- Clamped (C): W_i and U_{y_i} are penalized.

Where i is the node to be constrained

Because of the similarities, DS elements can be implemented in FEM codes.

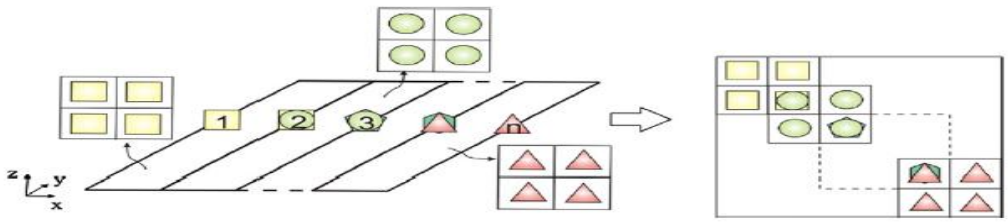

Fig. 5.Assembly of dynamic stiffness matrices

4.2. Wittrick–Williams algorithm

Once the global dynamic stiffness matrix of a structure is formed, the zeros of the determinant can be sought to determine the natural frequencies. This procedure can be hugely cumbersome because of the transcendental nature of the dynamic stiffness elements. Thus the plot of the frequency determinant can cause enormous difficulties. Such a plot can also miss coincident natural frequencies. The problem can be avoided by using the well known Wittrick and Williams's algorithm which guarantees that no natural frequencies of the structure are missed. The procedure is briefly summarized as follows.

A trial frequency w^* is chosen to compute the dynamic stiffness matrix$K^*$of the final structure which is then reduced to its upper triangular form by the usual Gauss elimination to obtain$K^{*\Delta}$. The number of negative terms on the leading diagonal of $K^{*\Delta}$was defined by Wittrick and Williamsas the sign count ***s(****K^*****)*** of the matrix. At this point, the number (***j***) of natural frequencies (***w***) which are lower than the trial frequency (w^*) is given by:

$$j = j_0 + s(K^*)$$

where$j_0$is the number of natural frequencies of single strip elements clamped on their opposite sides which are lower than the trial frequency. Assuming j_0 is known, a suitable procedure can be devised, for example the bi-section method, to bracket any natural frequency between an upper and lower bound to any desired accuracy. This allows quick and precise computations of the natural frequencies without numerically searching for the zeros of the determinant which may cause numerical difficulties due to the transcendental nature of stiffness element.

A drawback of the algorithm lays in computing the j_0 values.In this paper, $j_0$computation was avoided by using a sufficiently fine mesh to ensure $j_0 = 0$at the frequency range of interest. This was achieved by computing the first C–C natural frequency of the largest strip by splitting it into narrower strips, applying the DSM and subsequently compute the natural frequencies of the global structure that are below the C–C frequency computed earlier.

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V RESULTS

The procedure described above has been implemented in a MATLAB program to allow computation of natural frequencies of plates and plate assemblies. The first 11 natural frequencies of a simply supported (SS) square plate have been computed by the DSM theory of this paper based on the classical plate theory (CPT).

Natural frequencies (Hz)for orthotropic plate with various boundary conditions								
SS-F-SS-F			***SS-SS-SS-F***			***SS-C-SS-F***		
No.	*CPT*	*FEM*	*No.*	*CPT*	*FEM*	*No.*	*CPT*	*FEM*
1	11.421	11.357	1	19.169	19.161	1	14.806	14.723
2	14.615	14.498	2	40.165	40.140	2	22.490	22.675
3	26.448	26.329	3	65.812	65.793	3	41.816	41.611
4	35.310	35.059	4	70.517	70.015	4	71.167	70.660
5	47.658	47.558	5	83.056	82.882	5	90.720	90.530
6	65.261	65.176	6	108.882	108.74	6	105.207	105.07
7	76.791	76.555	7	111.363	111.02	7	110.169	109.83
8	92.656	92.524	8	150.344	150.13	8	130.295	130.11
9	104.658	104.53	9	156.479	156.24	9	156.609	156.37
10	108.905	108.90	10	200.37	200.37	10	166.386	166.31
11	114.052	114.04	11	200.482	200.39	11	212.452	212.48
SS-SS-SS-SS			***SS-C-SS-SS***			***SS-C-SS-C***		
No.	*CPT*	*FEM*	*No.*	*CPT*	*FEM*	*No.*	*CPT*	*FEM*
1	44.661	44.430	1	64.992	64.627	1	91.045	90.528
2	62.114	62.104	2	80.3659	79.192	2	102.827	102.00
3	91.596	91.147	3	105.567	105.05	3	124.56	123.95
4	132.454	131.51	4	142.147	142.05	4	159.245	158.11
5	160.056	159.72	5	193.035	192.63	5	205.441	205.01
6	177.872	177.64	6	199.901	199.64	6	244.750	244.43
7	173.786	183.22	7	216.01	215.35	7	259.037	258.24
8	207.535	207.24	8	242.265	241.92	8	264.867	264.49
9	246.69	246.32	9	254.61	254.23	9	282.396	281.99
10	248.659	248.28	10	280.147	279.72	10	316.914	316.43
11	300.798	300.66	11	328.148	327.67	11	362.7388	362.21

The plate considered is made up with a carbon-epoxy orthotropic material defined by its Young's moduli $E_x = 18.1 \times 10^9\ Pa$ and $E_y = 50.9 \times 10^9\ Pa$, itsPoisson's ratio $\vartheta_{xy} = 0.5$, its shear modulus $G_{xy} = 11 \times 10^9\ Pa$ and its mass density $\rho = 1526\ kg/m^3$. The $2a \times 2b$ dimensions of the plate are $1m \times 0.5\ m$, and its thickness is $0.002m$.

VI CONCLUSION

An exact dynamic stiffness method for an orthotropic plate with two opposite sides simply supported has been developed using classical plate theory. Explicit expressions for the terms of the dynamic stiffness matrix for all cases have been derived by extensive use of symbolic computation. The dynamic stiffness elements are assembled to investigate the free vibration behavior of complex structures following a procedure similar to that used in the finite element method. Once the dynamic stiffness matrix for the overall structure is formulated the eigenvalue problem is solved by using a modified version of the Wittrick and Williams algorithm.

The complete procedure starting from the development of the dynamic stiffness matrix and finishing with the calculation of natural frequencies has been implemented in a computer program using MATLAB. This enables computation of any number of exact natural frequencies of plates. Numerical results for a wide range of problems have been computed. These include prismatic plates with two opposite sides simply supported and the other two having any combination of boundary condition, such as simple support, clamped (built-in) support, or free edge.
The computed natural frequencies have been compared against results obtained by the finite element method with commercial FEA software application like Ansys 14. Very good convergence of both results was observed. Those in advantages of the DSM formulation presented are lower memory cost, higher precision and shorter computation time.

APPENDIX

Explicit expressions of the elements of the dynamic stiffness matrix are given as follows:

$$S_{vv} = (R_2 S_{h_2} r_1 r_2 (C_{h_1} - C_{h_2}))/\Delta - (C_{h_2} R_2 r_1 (S_{h_1} r_1 - S_{h_2} r_2))/\Delta - (R_1 S_{h_1} r_1 r_2 (C_{h_1} - C_{h_2}))/\Delta - (C_{h_1} R_1 (S_{h_2} {r_2}^2 - S_{h_1} r_1 r_2))/\Delta$$

$$S_{vm} = (C_{h_2} L_2 r_1 r_2 (C_{h_1} - C_{h_2}))/\Delta - (L_2 S_{h_2} r_1 (S_{h_1} r_1 - S_{h_2} r_2))/\Delta - (C_{h_1} L_1 r_1 r_2 (C_{h_1} - C_{h_2}))/\Delta - (L_1 S_{h_1} (S_{h_2} {r_2}^2 - S_{h_1} r_1 r_2))/\Delta$$

$$F_{vv} = (R_2 r_1 (C_{h_1} - C_{h_2}))/\Delta - (R_1 r_2 (C_{h_1} - C_{h_2}))/\Delta$$

$$F_{vv} = (L_2 (S_{h_2} r_1 - S_{h_1} r_2))/\Delta - (L_1 (S_{h_2} r_1 - S_{h_1} r_2))/\Delta$$

$$S_{vm} = (L_1 (C_{h_1} S_{h_2} r_1 - C_{h_2} S_{h_1} r_2))/\Delta - (L_2 (C_{h_1} S_{h_2} r_1 - C_{h_2} S_{h_1} r_2))/\Delta$$

$$F_{mm} = (L_2 (S_{h_2} r_1 - S_{h_1} r_2))/\Delta - (L_1 (S_{h_2} r_1 - S_{h_1} r_2))/\Delta$$

Where $R_1 = D_x r_{1_m}^3 - H\, r_{1_m} \alpha_m^2$, $R_2 = D_x r_{2_m}^3 - H\, r_{2_m} \alpha_m^2$,

$L_1 = -D_x r_{1_m}^2 - I\, \alpha_m^2$, $L_2 = -D_x r_{2_m}^2 - I\, \alpha_m^2$

$$\Delta = (C_{h_1}{}^2 r_1 r_2 - 2\,C_{h_1} C_{h_2} r_1 r_2 + C_{h_2}{}^2 r_1 r_2 - S_{h_2}{}^2 r_1 r_2 + S_{h_1} S_{h_2} r_1{}^2 + S_{h_1} S_{h_2} r_2{}^2 - S_{h_2}{}^2 r_1 r_2)$$

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