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FIXED POINT THEOREMS IN AN INTUITIONISTIC FUZZY METRIC SPACE

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ABSTRACT

In this paper, we prove a common fixed point theorem for a pair of mappings in an intuitionistic fuzzy metric space using the joint common limit in the range property of mappings called (JCLR) property. An example is also furnished which demonstrates the validity of main result. We also find an affirmative answer in Intuitionistic Fuzzy metric space to the problem of Rhoades [8].

Keywords: Intuitionistic fuzzy metric space, Common fixed point, noncompatible maps, R-weak commutativity of the mappings.

AMS Subject Classification (2000) : Primary 47H10, Secondary 54H25.

I. INTRODUCTION

The foundation of fuzzy Mathematics is laid by Lofti A. Zadeh [13] with the introduction of fuzzy sets in 1965, as a way to represent vagueness in everyday life. Atanassov [3] introduced and studied the concept of intuitionistic fuzzy set as a generalization of fuzzy sets. In 2004, Park [7] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et. al.[2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [5]. In 2006, Turkoglu [12] proved Jungck's [4] common fixed point theorem in the setting of intuitionistic fuzzy metric space for commuting mappings. Afterwards, many author proved common fixed point theorem using different variants in such spaces.

In 2002, Aamri and El-Moutawakil [1] defined the notion of (E.A) property for self mappings which contained the class of non-compatible mappings in metric spaces. It was pointed out that (E.A) property allows replacing the completeness requirement of the space with a more natural condition of closedness of the range as well as relaxes the complexness of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. Subsequently, there are a number of results proved for contraction mappings satisfying (E.A) property in fuzzy metric spaces. Most recently, Sintunavarat and Kumam [10] defined the notion of "common limit in the range" property (or

(CLR) property) in fuzzy metric spaces and improved the results of Mihet [6]. In [10], it is observed that the notion of (CLR) property never requires the condition of the closedness of the subspace while (E.A) property requires this condition for the existence of the fixed point (also see [11]). Many authors have proved common fixed point theorems in fuzzy metric spaces for different contractive conditions.

In this paper, our objective is to prove a common fixed point theorem for a pair of mappings in an intuitionistic fuzzy metric space using the joint common limit in the range property of mappings called (JCLR) property. An example is also furnished which demonstrates the validity of main result.

II. PRELIMINARIES

Definition 2.1. A binary operation $*$: $[0,1] \times [0,1] \to [0,1]$ is continuous t – norm if $*$ is satisfying the following conditions :

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0, 1]$;

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2. A binary operation $\lozenge$: $[0, 1] \times [0, 1] \to [0, 1]$ is continuous t – conorm if $\lozenge$ is satisfying the following conditions :

(i) $\lozenge$ is commutative and associative;

(ii) $\lozenge$ is continuous;

(iii) $a \lozenge 0 = a$ for all $a \in [0, 1]$;

(iv) $a \lozenge b \le c \lozenge d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [2] A 5 – tuple $(X, M, N, *, \lozenge)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t – norm, $\lozenge$ is a continuous t – conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions :

(i) $M(x, y, t) + N(x, y, t) \le 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t) \lozenge N(y, z, s) \ge N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all x, y in X :

The functions M(x, y, t) and N(x, y, t) denote the degree of nearness and the degree of non – nearness between x and y with respect to t, respectively.

Example 2.1. Let $X = \{\frac{1}{n} : n \in N\} \cup \{0\}$ with $*$ continuous t – norm and $\Diamond$ continuous t – conorm defined by a $*$ b = ab and a $\Diamond$ b = min {1, a+b} respectively, for all a, b $\in$ [0, 1]. For each t $\in$ (0, ∞) and x, y $\in$X, define M and N by

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & t > 0, \\ 1 & t = 0. \end{cases}$$

Then, (X, M, N, $*$, $\Diamond$) is an intuitionistic fuzzy metric space, (for k = 1 [8]).

Definition 2.4. Let A and B be two self mappings of an intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$). Then the maps A and B are said to be compatible if for all t > 0,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z $\in$ X.

Definition 2.5. Self maps A and S of an intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$) are said to be R-weakly commuting if there exists some real number R such that

M(ASx, SAx, t) ≥ M(Ax, Sx, t/R) and N(ASx, SAx, t) ≤ N(Ax, Sx, t/R) for each x $\in$ X and t > 0.

Definition 2.6. Self maps A and S of an intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$) are said to be R-weaklycommuting of type (A_g) if there exists some real number R such that

M(AAx, SAx, t) ≥ M(Ax, Sx, t/R) and N(AAx, SAx, t) ≤ N(Ax, Sx, t/R) for each x $\in$ X and t > 0.

Definition 2.7. Let A and B be two self mappings of an intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$). Then the maps A and B satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some

z $\in$ X.

Definition 2.8. Let f and g be two self mappings of an intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$). Then the maps f and g satisfy the "common limit in the range of g" property (shortly, (CLRg) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu$ for some u $\in$ X.

Remark 2.1. From definition 2.4, it is inferred that two maps A and B of an intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$) are non-compatible if and only if there exists atleast one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z $\in$ X, but for some t > 0 either $\lim_{n\to\infty} M(ABx_n, BAx_n, t) \neq 1$ and $\lim_{n\to\infty} N(ABx_n, BAx_n, t) \neq 0$ or the limit does not exist.

III. MAIN RESULTS

Theorem 3.1. Let f and g be pointwise R-weakly commuting self mappings of an Intuitionistic Fuzzy metric space (X, M, N, ∗, ◊) satisfying the property (CLRg) and :

(3.1) $f(X) \subset g(X)$

(3.2) $M(fx, fy, kt) \geq M(gx, gy, t)$ and $N(fx, fy, kt) \leq N(gx, gy, t)$, $k \geq 0$,

(3.3) $M(fx, f^2x, t) > \max\{M(gx, gfx, t), M(fx, gx,t), M(f^2x, gfx, t), M(fx, gfx, t), M(gx, f^2x, t)\}$ and
$N(fx, f^2x, t) < \min\{N(gx, gfx, t), N(fx, gx,t), N(f^2x, gfx, t), N(fx, gfx, t), N(gx, f^2x, t)\}$,
whenever $fx \neq f^2x$.

If the range of f or g is a complete subspace of X, then f and g have a common fixed point.

Proof. Since f and g are satisfy the property (CLRg), then there exists a sequence $\{x_n\}$ such that $fx_n \to gp$ and $gx_n \to gp$, for some $p \in X$ as $n \to \infty$.

Put $x = x_n$ and $y = p$ in (3.2), we get

$$M(fx_n, fp, kt) \geq M(gx_n, gp, t) \text{ and } N(fx_n, fp, kt) \leq N(gx_n, gp, t).$$

Letting $n \to \infty$,

$$M(gp, fp, kt) \geq M(gp, gu, t) \text{ and } N(gp, fu, kt) \leq N(gp, gu, t).$$

Hence fu = gu.

Since f and g are R-weak commuting, there exists R > 0 such that

$$M(fgu, gfu, t) \geq (M(fu, gu, t/R) = 1 \text{ and } N(fgu, gfu, t) \leq N(fu, gu, t/R) = 0,$$

that is, fgu = gfu and ffu = gfu = ggu. If fu ≠ ffu, using (3.3),we get

$$M(fu, ffu, t) > \max\{M(gu, gfu, t), M(fu, gu, t), M(ffu, gfu, t), M(ffu, gfu, t), M(gu, ffu, t)\}= M(fu, ffu, t)$$

and

$$N(fu, ffu, t) < \min\{N(gu, gfu, t), N(fu, gu, t), N(ffu, gfu, t), N(ffu, gfu, t), N(gu, ffu, t)\} = N(fu, ffu, t),$$

a contradiction.

Hence, fu = ffu and fu = ffu = fgu = gfu = ggu.

Hence fu is a common fixed point of f and g. The case when f(X) is a complete subspace of X is similar to the above case since $f(X) \subset g(X)$. Hence we have the theorem.

We now give an example to illustrate the above theorem.

Example 3.1. Let X = [2, 20] and and d be the usual metric on X. For each $t \in [0, \infty)$, define (M, N) for x, y $\in$ X by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t>0, \\ 0 & t=0 \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t>0, \\ 1 & t=0. \end{cases}$$

Clearly (X, M, N, ∗, ◊) is an Intuitionistic Fuzzy metric space. Define f, g : $X \to X$ as

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$$f(x) = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 6, & \text{if } 2 < x \leq 5 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & \text{if } x = 2 \\ x + 4, & \text{if } 2 < x \leq 5 \\ \frac{4x+10}{15}, & \text{if } x > 5. \end{cases}$$

Clearly, f and g satisfy all the conditions of Theorem 3.1 and have common fixed point at x = 2.

Also, $f(X) \subset g(X)$ and f and g are pointwise R-weakly commuting mappings and satisfy the (CLRg) property.

Setting k = 1 in the above theorem, we get the following theorem :

Theorem 3.2. Let f and g be pointwise R-weakly commuting self mappings of a intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ satisfying the property (CLRg) and :

(3.4) $f(X) \subset g(X)$

(3.5) $M(fx, fy, t) \geq M(gx, gy, t)$ and $N(fx, fy, t) \leq N(gx, gy, t)$,

(3.6) $M(fx, f^2x, t) > \max\{M(gx, gfx, t), M(fx, gx,t), M(f^2x, gfx, t), M(fx, gfx, t), M(gx, f^2x, t)\}$ and

$N(fx, f^2x, t) < \min\{N(gx, gfx, t), N(fx, gx,t), N(f^2x, gfx, t), N(fx, gfx, t), N(gx, f^2x, t)\}$,

whenever $fx \neq f^2x$.

If the range of f or g is a complete subspace of X, then f and g have a common fixed point.

Theorem 3.1 has been proved by using the concept of (CLRg) property which has been introduced in a recent work by Sintunavarat and Kumam [10]. They have shown that (CLRg) property is more general than the notion of non-compatibility. It may, however, be observed that by using the notion of non-compatible maps in place of (CLRg) property, we cannot only prove a Theorem 3.1 above, but, in addition, we are able to show also that maps are discontinuous at their common fixed points. We do this in our next theorem and thus find out an answer in Intuitionistic Fuzzy metric space to the problem of Rhoades [8].

Theorem 3.3. Let f and g be noncompatible pointwise R-weakly commuting self mappings of type (A_g) of an Intuitionistic Fuzzy metric space $(X, M, N, *, \lozenge)$ satisfying :

(3.7) $f(X) \subset g(X)$

(3.8) $M(fx, fy, kt) \geq M(gx, gy, t)$ and $N(fx, fy, kt) \leq N(gx, gy, t)$, $k \geq 0$,

(3.9) $M(fx, f^2x, t) > \max\{M(gx, gfx, t), M(fx, gx,t), M(f^2x, gfx, t),$
$M(fx, gfx, t), M(gx, f^2x, t)\}$ and

$N(fx, f^2x, t) < \min\{N(gx, gfx, t), N(fx, gx,t), N(f^2x, gfx, t),$
$N(fx, gfx, t), N(gx, f^2x, t)\}$, whenever $fx \neq f^2x$.

If the range of f or g is a complete subspace of X, then f and g have a common fixed point and the fixed point is the point of discontinuity.

Proof. Since f and g are non-compatible maps, then there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = p \text{ and } \lim_{n\to\infty} gx_n = p, \text{ for some } p \in X, \qquad (1)$$

but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist.

Since $p \in f(X)$ and $f(X) \subset g(X)$, there exists some point u in X such that p = gu, where $p = \lim_{n\to\infty} gx_n$. If $fu \neq gu$, then

$$M(fx_n, fu, kt) \geq M(gx_n, gu, t) \text{ and } N(fx_n, fu, kt) \leq N(gx_n, gu, t).$$

Letting $n \to \infty$,

$$M(gu, fu, kt) \geq (M(gu, gu, t) \text{ and } N(gu, fu, kt) \leq N(gu, gu, t)$$

Hence fu = gu.

Since f and g are R-weak commuting of type (A_g), there exists R > 0 such that

$$M(ffu, gfu, t) \geq (M(fu, gu, t/R) = 1 \text{ and } N(ffu, gfu, t) \leq N(fu, gu, t/R) = 0,$$

that is, fu = gfu and ffu = fgu = gfu = ggu.

If $fu \neq ffu$, using (3.9),we get

$$M(fu, ffu, t) > \max\{M(gu, gfu, t), M(fu, gu, t), M(ffu, gfu, t), M(ffu, gfu, t), M(gu, ffu, t)\}$$
$$= M(fu, ffu, t)$$

and

$$N(fu, ffu, t) < \min\{N(gu, gfu, t), N(fu, gu, t), N(ffu, gfu, t), N(ffu, gfu, t), N(gu, ffu, t)\}$$
$$= N(fu, ffu, t),$$

a contradiction.

Hence, fu = ffu and fu = ffu = fgu = gfu = ggu.

Hence fu is a common fixed point of f and g. The case when f(X) is a complete subspace of X is similar to the above case since $f(X) \subset g(X)$. We now show that f and g are discontinuous at the common fixed point p = fu = gu. If possible, suppose f is continuous. Then considering the sequence $\{x_n\}$ of (1) we get $\lim_{n\to\infty} ffx_n = fp = p$. R-weak commutativity of type (A_g) implies that $M(ffx_n, gfx_n, t) \geq M(fx_n, gx_n, t/R) = 1$ and $N(ffx_n, gfx_n, t) \leq N(fx_n, gx_n, t/R) = 0$, which on letting $n \to \infty$ this yields $\lim_{n\to\infty} gfx_n = fp = p$. This, in turn, yields $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ and

$\lim_{n\to\infty} N(fgx_n, gfx_n, t) = 0$. This contradicts the fact that $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either nonzero or nonexistent for the sequence $\{x_n\}$ of (1). Hence f is discontinuous at fixed point. Similarly, g is also discontinuous at fixed point.

Thus, both f and g are discontinuous at their common fixed point. Hence we have the theorem.

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