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COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS SATISFYING CONTRACTIVE CONDITION OF INTEGRAL TYPE IN CONE METRIC SPACE

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ABSTRACT

In this paper, we prove a common fixed point theorem for compatible mappings satisfying general contractive condition of integral type in cone metric space. Our result extand and generalize some results of Huang and Zhang [10], Khojasteh et.al.[14], Badshah and Pariya [6] and others.

Keywords. Contractive Condition of Integral Type,Cone Metric Space, Weakly Compatible Mappings, Common Fixed Point. hematics Subject Classification. Primary 47H10, Secondary 54H25.

I. INTRODUCTION AND PRELIMINARIES

Huang and Zhang [10] introduced the notion of cone metric space and proved some fixed point theorems in cone metric spaces for mapping satisfying various contractive conditions. Many authors study this subject and many fixed point theorems are proved. For example [2,11,17,9]. Jungck [12] gave a common fixed point theorem for commuting mappings, which generalizes the Banach's fixed point theorem and he also introduced the concept of compatible maps which is weaker than weakly commuting maps. Branciari [17] obetained a fixed point result for a single mapping satisfying Banach's contraction principle for an integral type inequality. Further this result was generalized by [3,4,5,1,8,16,15,19].

The following definitions are due to Huang and Zhang [10].

Definition 1.1. Let E be a real Banach space and P be a subset of E. P is called a cone if;

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a, b \in R$, $a\, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(c) $x \in P$ and $-x \in P \Rightarrow x = 0$

Given a cone $P \subseteq E$, we define a partial ordering "$\leq$" with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $y - x \in P^o$, where P^o stands for the interior of P.

The cone P is called normal if there is a number K > 0 such that for all $x, y \in E$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|$$

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The least positive number satisfying above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

For some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose E is a Banach space, P is a cone in E with $P \neq \emptyset$ and $\leq$is partial ordering with respect to P.

Definition 1.2. A cone metric space is an ordered pair (X,d), where X is any set and $d : X \times X \to E$ is a mapping satisfying :

(a) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Definition 1.3. Let (X,d) be a cone metric space $\{x_n\}$ a sequence in X and $x \in X$.If for any $c \in E$ with $c >> 0$, there is N such that for all n>N, $d(x_n, x) << c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converge to x. i.e $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.4. Let (X,d) be a cone metric space $\{x_n\}$ a sequence in X, if for any $c \in E$ with $c>>0$, there is N such that for all n, m >N, $d(x_m, x_n) << c$,then $\{x_n\}$ is called Cauchy sequence inX. **Lemma 1.1[18].** Let (X,d) be a cone metric space, P a normal cone with a normal constant K. Let$\{x_n\}$ and $\{y_n\}$be two sequences in X and $y_n \to y, x_n \to x$ as $n \to \infty$,

then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Lemma 1.2[18]. Let (X,d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converge to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

Lemma 1.3[18]. Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Definition 1.5 [6]. Two self-maps f and g of a set X are compatible if,$\lim_{n\to\infty}(fg\,x_n, gf\,x_n) = 0$ whenever $\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n =$t for some $t \in X$.

Definition 1.6 [14]. The function $\phi: P \to E$ is called subadditivecone integrable function if and only if for all $a,b \in P$

$$\int_0^{a+b} \phi dp \leq \int_0^a \phi dp + \int_0^b \phi dp.$$

Lemma 1.4[14]. If $[a,b]\subseteq[c,d]$,then

(1)$\int_a^b f dp \leq \int_c^d f dp$, for $f \in L^1(X, P)$.

(2)$\int_a^b (\alpha f + \beta g) dp \leq \alpha \int_a^b f dp + \beta \int_a^b f dp$,for $f, g \in L^1(X, P)$ and $\alpha, \beta \in \mathbb{R}$.

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II. MAIN RESULT

In this section we extand and generalize some results of Huang and Zhang [10], Khojasteh et.al.[14], Badshah and Pariya [6] and others. Also we prove a common fixed point theorem for compatible mappings satisfying general contractive condition of integral type in cone metric space.

Theorem 2.1. Let (X,d) be a complete cone metric space and P a normal cone. Suppose that ϕ:P→P is a nonvanishing map and a subadditive cone integrable on each [a,b]⊂ P such that for each $\varepsilon >> 0$, $\int_0^{\varepsilon} \phi(t)dt >> 0$ .If T,S : X→X are compatible maps such that $T(X) \subset S(X)$,for all $x, y \in X$ and satisfying,

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \psi(\int_0^{M(x,y)} \phi(t)dt) \qquad (1)$$

whereM(x,y)= $\alpha \frac{(d(Sx,Tx))^2+(d(Sy,Ty))^2}{d(Sx,Tx)+d(Sy,Ty)} + \beta\, d(Sx,Sy)$and $\alpha, \beta \geq 0$, $2\alpha + \beta < 1$and $\psi \in (0,1)$. Then T and S have unique common fixed point.

Proof. Fix x∊ X. Let {y_n} be defined by $y_n = Tx_n = Sx_{n+1,}$ for all $n \geq 0$.

If $y_n = y_{n+1}$ for any n, then for all$m > n$, hence {y_n} is Cauchy sequence.

If $y_n \neq y_{n+1}$ for any n, then by (1), we have

$$\int_0^{d(y_n,y_{n+1})} \phi(t)dt= \int_0^{d(Tx_n,Tx_{n+1})} \phi(t)dt$$

$$\leq \psi(\int_0^{M(x_n,x_{n+1})} \phi(t)\,dt) \qquad (2)$$

where,

$$M\,(x_{n,}x_{n+1}) \leq \alpha \frac{(d(Sx_n,Tx_n))^2+(d(Sx_{n+1},Tx_{n+1}))^2}{d(Sx_n,Tx_n)+d(Sx_{n+1},Tx_{n+1})} + \beta d(Sx_n, Sx_{n+1})$$

$$\leq \alpha \frac{(d(y_{n-1},y_n))^2+(d(y_n,y_{n+1}))^2}{d(y_{n-1},y_n)+d(y_n,y_{n+1})} + \beta d(y_{n-1},y_n)$$

$$\leq \alpha(d(y_{n-1},y_n) + d(y_n, y_{n+1})) + \beta d(y_{n-1},y_n)$$

i.e. $d(y_n, y_{n+1}) \leq (\alpha + \beta)\, d(y_{n-1},y_n) + \alpha d(y_n, y_{n+1})$

$$d(y_n, y_{n+1}) \leq \frac{\alpha+\beta}{1-\alpha} d(y_{n-1},y_n)$$

$d(y_n, y_{n+1}) \leq hd(y_{n-1},y_n)$, *where* $h = \frac{\alpha+\beta}{1-\alpha} < 1$

Hence by (2), $\int_0^{d(y_n,y_{n+1})} \phi(t)dt \leq \psi(\int_0^{h.d(y_{n-1},y_n)} \phi(t)\,dt)$ *(3)*

By inductivity, we obtain,

$$\int_0^{d(y_n,y_{n+1})} \phi(t)dt \leq \psi\left(\int_0^{h.d(y_{n-1},y_n)} \phi(t)\,dt\right) \leq \psi\left(\int_0^{h.h.d(y_{n-2},y_{n-1})} \phi(t)\,dt\right)$$

$$\leq \cdots \leq \psi(\int_0^{h^n.d(y_0,y_1)} \phi(t)\,dt)\,i.e.\ \int_0^{d(y_n,y_{n+1})} \phi(t)dt \leq \psi\left(\int_0^{h^n.d(y_0,y_1)} \phi(t)\,dt\right) \qquad (4)$$

By triangle ineqeality of a cone metric for m>n we get

$$\int_0^{d(y_n,y_m)} \phi(t)dt = \int_0^{d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\cdots+d(y_{m-1},y_m)} \phi(t)dt$$

$$\leq \psi\left(\int_0^{h^n d(y_0,y_1)+h^{n-1}d(y_0,y_1)+\cdots+h^{m-1}d(y_0,y_1)} \phi(t)\,dt\right)$$

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$\leq \psi(\int_0^{\frac{h^n}{1-h}d(y_0,y_1)} \phi(t)\,dt)$

Since, $0 \leq h < 1$,*by normality of cone,*

$$\left\|\int_0^{d(y_n,y_m)} \phi(t)dt\right\| \leq K\left\|\psi\left(\int_0^{\frac{h^n}{1-h}d(y_0,y_1)} \phi(t)\,dt\right)\right\| \to 0$$

Hence by Lemma1.3, $\{y_n\}$is a Cauchy sequence. Since (X,d) is a complete cone metric space and since $y_n = Tx_n = Sx_{n+1}$ in $T(X)$, and$\{y_n\}$is Cauchy sequence in T(X), so it must be convergent in T(X), then there exists $y_0 \in X$ such that

$$\lim_{n\to\infty} y_n = y_0 \qquad (5)$$

Note that it is also true for S(X) since $y_0 \in T(X) \subset S(X)$.

Also, $\lim_{n\to\infty} Tx_n = y_0$and $\lim_{n\to\infty} S_{n+1} = y_0$

Now since S and T are compatible, then we have

$$y_0 = Ty_0 = TSy_0 = STy_0 = Sy_0 = y_0$$

Hence y_0 is commom fixed point of S and T.

For uniquencess, assume $x,y \in X$and $x \neq y$ are two common fixed points of S and T.

Then by (1),

$$\int_0^{d(x,y)} \phi(t)dt = \int_0^{d(Tx,Ty)} \phi(t)dt \leq \int_0^{M(x,y)} \phi(t)dt \qquad (6)$$

Where,

M(x,y)= $\alpha\frac{(d(Sx,Tx))^2+(d(Sy,Ty))^2}{d(Sx,Tx)+d(Sy,Ty)} + \beta d(Sx,Sy)$

$\leq \alpha\, d(Sx,Tx) + \alpha d(Sy,Ty) + \beta d(Sx,Sy)$

$= \alpha d(x,x) + \alpha d(y,y) + \beta d(x,y)$

$= \beta d(x,y)$

i.e.$\int_0^{d(x,y)} \phi(t)dt \leq \psi(\int_0^{\beta d(x,y)} \phi(t)dt)$, which is contradiction, since $\psi \in (0,1)$.

Hence S and Thave unique common fixed point. This completes the proof.

Corollary 2.1 Let (X,d) be a complete cone metric space and P a normal cone. Suppose that ϕ:P→P is a nonvanishing map and a subadditive cone integrable on each [a,b]⊂ P such that for each $\varepsilon >> 0$,$\int_0^{\varepsilon} \phi(t)dt >> 0$ .If T,S : X→X are weakly compatible maps such that $T(X) \subset S(X)$,for all $x,y \in X$ and condition (1) is satisfied. Then T and S have unique common fixed point.

Proof. Since weakly compatible maps are compatible. Therefore the result follows from theorem 2.1.

Corollary 2.2 Let (X,d) be a complete cone metric space and P a normal cone. Suppose that ϕ:P→P is a nonvanishing map and a subadditive cone integrable on each [a,b]⊂ P such that for each $\varepsilon >> 0$,$\int_0^{\varepsilon} \phi(t)dt >> 0$ .If T,S : X→X are occasionally weakly compatible maps such that $T(X) \subset S(X)$,for all $x,y \in X$ and condition (1) is satisfied. Then T and S have unique common fixed point.

Proof. Since occasionally weakly compatible maps are weakly compatible. Therefore the result follows from theorem 2.1.

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