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DOMINATING χ-COLOR NUMBER OF GENERALIZED PETERSEN GRAPHS

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ABSTRACT

Dominating χ-color number of a graph $G$is defined as the maximum number of color classes which are dominating sets of $G$and is denoted by d_χ, where the maximum is taken over all χ-coloring of G. In this paper, we discussed the dominating χ-color number of Generalized Petersen Graphs. We have also discussed the condition under which chromatic number equals dominating χ-color number of Generalized Petersen Graphs.

Keywords: Chromatic number, Dominating number, Dominating χ-Color number, Generalized Petersen Graph, Strong Dominating χ-Color number.

I. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple, connected, finite, undirected graph. The order and size of G are denoted by n and m [1]. A set $D \subseteq V(G)$ is a dominating set of G, if for every vertex $x \in V(G) \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$. And the set D is said to be strong dominating set of G, if it satisfy the additional condition $d(x, G) \leq d(y, G)$[2]. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. Also D is said to be weak dominating set of G if it satisfy the additional condition $d(u, G) \geq d(v, G)$. The weak domination number $\gamma_w(G)$ is defined as the minimum cardinality of a weak dominating set. It was introduced by Sampathkumar and PushpaLatha (Discrete Math. 161 (1996)235-242)[3].

The generalized Petersen graph $GP(n, k)$, also denoted $P(n, k)$ (Biggs 1993, p. 119; Pemmaraju and Skiena 2003, p. 215), for n ≥3 and $1 \leq k < n/2$ is a graph consisting of an inner star polygon (n, k)(Circulant Graph) and an outer regular polygon (n) (cycle graph C_n) with corresponding vertices in the inner and outer polygons connected with edges. $GP(n, k)$has $2n$ nodes and $3n$ edges. These graphs were introduced by Coxeter (1950) named by Watkins (1969) and around 1970 popularized by Frucht, Graver and Watkins. [4, 5, 6].

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II. PRELIMNARY RESULT

Definition 2.1: [7] Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2, \dots, V_k$ be a k −coloring of G. Let d_C denote the number of color classes in C which are dominating sets of G. Then $d_\chi(G) = max_c d_C$ where the maximum is taken over all the k −colorings of G, is called the dominating χ-color number of G.

Definition 2.2:[8] Let G be a graph with $\chi(G) = k$. Let $C = V_1, V_2, \dots, V_k$ be a k −coloring of G. Let d_C denote the number of color classes in C which are strong dominating sets of G. Then $d_\chi(G) = max_c d_C$ where the maximum is taken over all the k −colorings of G, is called the strong dominating χ-color number of G.

And strong dominating χ-color number of G is denoted by $sd_\chi(G)$.Strong-dominating χ-color number $sd_\chi(G)$ exists for all graphs G and $1 \leq sd_\chi(G) \leq d_\chi(G) \leq \chi(G)$.

Definition 2.3: [9] The Generalized Petersen graph $P(n, k)$ is a graph with vertex and edge set given by, $V(P(n,k)) = \{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$

$E(P(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, \dots, n-1\}$

Where the subscripts are expressed as integers modulo n $(n \geq 5)$.

Propositions: [10]

1. $sd_\chi(G) = d_\chi(G) = \begin{cases} 3 & n \text{ is odd multiple of } 3 \\ 2 & \text{otherwise} \end{cases}$
2. $P(n, k)$ is a 3-regular graph with $2n$ vertices and $3n$ edges.
3. $P(n, k)$ is bipartite if and only if n is even and k is odd.
4. If G is regular, then $sd_\chi(G) = d_\chi(G)$

III. DOMINATING χ-COLOR NUMBER OF GENERALIZED PETERSEN GRAPHS

Let $G = P(n, k)$ where $k < \frac{n}{2}$, be the Generalized Petersen Graph with $2n$ vertices and $3n$ edges. By the definition of Generalized Petersen Graph, the set of outer vertices, say U and the set of inner vertices, say V are labeled $u_0, u_1, \dots u_{n-1}$ and $v_0, v_1, \dots v_{n-1}$ respectively. For any $k < \frac{n}{2}$, the vertices $u_0, u_1, \dots u_{n-1}$ are adjacent to $u_1, u_2, \dots u_0$ respectively. By the construction of $P(n, k)$ the induced sub-graph of U will form a cycle of length n. But the adjacency of the vertices of V is related to the common factors of n and k. Since $P(n, k)$ is 3-regular graph, $d_\chi(P(n,k)) = sd_\chi(P(n,k))$.

Lemma 3.1 Let G be a graph with vertex set $\{v_0, v_1, \dots, v_{n-1}\}$ and the edge set $\{v_i v_{i+k} : i = 0, \dots, n-1\}$ where the subscripts are expressed as integers modulo n $(n \geq 5)$. For any $k < \frac{n}{2}$, if $gcd(n,k) = 1$, then $G \cong C_n$ and if $gcd(n,k) = g$, then $G \cong g C_{n/g}$

Lemma 3.2 If $G = P(n,k)$ is a generalized Petersen graph, then $2 \leq \chi(G) \leq 3$

Theorem 3.3 If $G = P(n,k)$ is a generalized Petersen graph, then $2 \leq d_\chi(G) \leq 3$

Proof

By Lemma 3.1, the induced sub graphs of U and $V$are isomorphic to cycle $C_n$and $g = \gcd(n, k)$copies of $C_{n/g}$that has dominating χ −color number is either 2 or 3. By Lemma 3.2, $\chi(G)$ is either 2 or 3, that means there exist a coloring function c from $U \cup V$ to {0,1,2} such that u_i and $v_i$does not have the same color. So, adding the edges $u_i v_i$, $i = 0,1, \dots, n-1$ to connect the induced sub-graphs U and V to get $P(n, k)$will not reduce the dominating χ −color number. Thus $d_\chi(G) = 2$ or 3.

Theorem 3.4 Let $P(n, k)$be a generalized Petersen graph. Then $d_\chi(P(n, k)) = 2$ if either $P(n, k)$ is bipartite or n is not a multiple of 3

Proof

If $P(n, k)$is bipartite, then each partition have different color and dominates one another. Since both color class are dominating set, $d_\chi(P(n, k)) = 2$.

If n is not multiple of 3, Now the set of outer vertices, say U and the set of inner vertices, say V are labeled $u_0, u_1, \dots u_{n-1}$ and $v_0, v_1, \dots v_{n-1}$ respectively, as per the definition.

By Lemma− 3.2, the chromatic number of $P(n, k)$is either 2or3. Since n and $\frac{n}{g}$ is not multiple of 3, $d_\chi(C_n) = 2$and $d_\chi(C_{n/g}) = 2$ .There exist a coloring function c from $U \cup V = \{u_i, v_i, 0 \le i \le n-1\}$ to $\{0,1,2\}$ such that u_i and v_i does not have the same color. Since n is not multiple of 3, adding the edges $u_i v_i$, $i = 0,1,2, \dots, n-1$ to connect the induced sub-graphs U and V to get $P(n, k)$which makes even cycles $u_i v_i v_{i+g} u_{i+g} u_{i+g-1} \dots u_{i-1} u_i$, for $i = 0,1,2, \dots, n-1$, will not affect the dominating χ −color number $d_\chi(G) = 2$.

Theorem 3.5 Let $P(n, k)$be a generalized Petersen graph. Then$d_\chi(P(n, k)) = 3$ if $n \equiv 3 (mod\ 6)$or $n \equiv 0 (mod\ 6)$ and $k \equiv 0\ (mod\ 2)$

Proof

Case A: If$n \equiv 3 (mod\ 6)$

If $\gcd(n, k) = 1$and n is odd, by Lemma− 3.1, the induced sub-graph of outer vertices U of $G \cong C_n$. And also, the induced sub-graph of inner vertices V of $G \cong C_n$. Here, C_n is a cycle of odd length. Since n is odd and multiple of 3, by Proposition−1, dominatingχ −color number $d_\chi$of odd cycle C_n is 3.

If $\gcd(n, k) = g$and n is odd, g cannot be even. Also it is clear that n/g is odd. By Lemma− 3.1, the outer vertices U of$G \cong C_n$. And the induced sub-graph of inner vertices Vwill form g numbers of disjoint cycles $C_{n/g}$. Since n and n/g is odd multiple of 3, $\chi(C_n) = 3$ and $\chi(C_{n/g}) = 3$.

Since $\chi(G) = 3$, there exists a coloring function c from $U \cup V = \{u_i, v_i, 0 \le i \le n-1\}$ to $\{0,1,2\}$ such that u_i and v_i does not have the same color. So, adding the edges $u_i v_i$, $i = 0,1,2, \dots, n-1$ to connect the induced sub-graphs U and V to get $P(n, k)$which makes cycles$u_i v_i v_{i+g} u_{i+g} u_{i+g-1} \dots u_{i-1} u_i$, for $i = 0,1,2, \dots, n-1$, will not reduce the dominating χ −color number $d_\chi(G) = 3$.

Case B: If $n \equiv 0 (mod\ 6)$ and $k \equiv 0\ (mod\ 2)$

Since n and k is even, $\gcd(n, k) > 1$.Let g be $\gcd(n, k)$. Also$g$ is even.

If n/g is odd. By Lemma – 3.1, the outer vertices U of $G \cong C_n$. The induced sub-graph of inner vertices V will form g numbers of disjoint cycles $C_{n/g}$.

If n/g is odd multiple of 3, then $\chi(C_n) = 2$ and $\chi(C_{n/g}) = 3$.

Also $d_\chi(C_n) = 2$ and $d_\chi(C_{n/g}) = 3$. So the chromatic number of a graph with vertex $U \cup V$ is 3. If n/g is even multiple of 3, then $\chi(C_n) = 2$ and $\chi(C_{n/g}) = 2$.

Also $d_\chi(C_n) = 2$ and $d_\chi(C_{n/g}) = 2$. Also $\chi(C_n) = 2$ and $\chi(C_{n/g}) = 2$. Adding the edges $u_i v_i$, $i = 0,1,2,\dots,n-1$ to connect the induced sub-graphs U and V to get $P(n,k)$ which makes odd cycles $u_i v_i v_{i+g} u_{i+g} u_{i+g-1} \dots u_{i-1} u_i$, for $i = 0,1,2,\dots,n-1$, increase the chromatic number. So the chromatic number of a graph with vertex $U \cup V$ is 3.

In both cases, there exist a coloring function c from $U \cup V = \{u_i, v_i, 0 \le i \le n-1\}$ to $\{0,1,2\}$ such that,

For $i \equiv p (mod\ 3g)$,

$$c(v_i) = j \text{ if } \left\lfloor \frac{i}{g} \right\rfloor \equiv j\ (mod\ 3) \text{ and } c(u_i) = \begin{cases} 1 & \text{if } 0 \le p < g \text{ and even} \\ 2 & \text{if } 0 \le p < 2g \text{ and odd} \\ 0 & \text{if } g \le p < 2g \text{ and even} \\ 1 & \text{if } 2g \le p < 3g \text{ and even} \\ 0 & \text{if } 2g \le p < 3g \text{ and odd} \end{cases}$$

where c is expressed as integer modulo 3.

Hence the dominating χ -color number $d_\chi(G) = 3$.

Theorem 3.7 Let $G = P(n,k)$ be a generalized Petersen graph with $k = 1$ and $n \equiv 0\ (mod\ 3)$, then $d_\chi(G) = \chi(G)$.

Proof

If n is even multiple of 3, then it is clear from the Proposition-3, $P(3m,1)$ is bipartite. Hence $d_\chi(G) = \chi(G) = 2$. If n is odd multiple of 3, then there exist coloring function c from the set of vertices $\{u_i, v_i, 0 \le i \le n-1\}$ to {0, 1, 2} is defined by $c(u_i) = j$ if $i \equiv j\ (mod\ 3)$ and $c(v_i) = j+1$ if $i \equiv j\ (mod\ 3)$ where c is expressed as integers modulo 3. Clearly, the each color class $C[0], C[1], C[2]$ dominates $P(n,1)$. Hence $d_\chi(G) = \chi(G) = 3$.

REFERENCES

[1] Douglas B. West, Introduction to Graph Theory, 2nd edition Prentice Hall, Inc, Mass, 2001.

[2] Dieter Rautenbach, Bounds on the strong domination number, Discrete Mathematics, 215, 2000, 201-212.

[3] E. Sampathkumar and L. PushpaLatha, Strong, weak Domination and Domination Balance in a Graph, Discrete Math. 161 (1996) 235-242.

[4] ArjanaZitnik , Boris Horvat, TomazPisanski, All generalized Petersen graphs are unit-distance graphs. Preprint series, IMFM, ISSN 2232-2094, no. 1109, January 4, 2010.

[5] H. S. M. Coxeter, Self-dual configurations and regular graphs, Bull. Amer. Math. Soc. 56 (1950), 413-455.

[6] M.E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, J. Combin. Theory, 6 (1969), 152-164.

[7] S. Arumugam, I. SahulHamid and A. Muthukamatchi, Independent Domination and Graph Colorings, In: Proceedings of international Conference on Discrete Mathematics. Lecture Note series in Mathematics, Ramanujan Mathematical Society (India)(2008).

[8] T.Ramachandran, A.Naseer Ahmed, Strong and Weak Dominating χ-Color Number of K-Partite Graph, International Journal of Advanced Technology in Engineering and Science, Vol 02, Issue 09, 2014.

[9] B. Alspach, The classification of Hamiltonian generalized Petersen graphs, J. Combin. Th. (B), 34 (1983), 293- 312.

[10] Fox.J, R. Gera, and P. Stanica, The Independence Number for the Generalized Petersen Graphs, Ars Combin. 103 (2012) 439–451.