A COMMON FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN FUZZY METRIC SPACE

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ABSTRACT

In this paper, we prove a common fixed point theorem for occasionally weakly compatible mappings in fuzzy metric spaces using the property (E.A.).

Keywords : Compatible Mappings, Occasionally Weakly Compatible Mappings and Common Fixed Point

I. INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh [21] in 1965. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [11], George and Veeramani [8] modified the notion of fuzzy metric space with the help of continuous t-norms.

For example, Deng [5], Ereeg [6], Fang [7], George and Veeramani [8], Kaleva and Seikkala [12], Kramosil and Michalek [11] have introduced the concept of fuzzy metric spaces in different ways. In applications of fuzzy set theory the field of engineering has undoubtedly been a leader. All engineering disciplines such as civil engineering, electrical engineering, nuclear engineering etc. have already been affected to various degrees by the new methodological possibilities opened by fuzzy sets.

II. PRELIMINARIES

Definition 2.1. [16] A binary operation $*: [0,1]^2 \to [0, 1]$ is called a continuous t-norm if ([0, 1],*) is an abelian topological monoid; i.e.

(1) * is associative and commutative,

(2) * is continuous,

(3) $a*1=a$ for all $a \in [0,1]$,

(4) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for each $a,b,c,d \in [0,1]$.

Two typical examples of a continuous t-norm are $a * b=ab$ and $a * b=\min\{a, b\}$.

Definition 2.2. [15**]** The 3-tuple (X, M,*) is called a fuzzy metric space if X is an arbitrary non-empty set,* is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions.

for each $x,y,z \in X$ and $t,s > 0$,

(FM-1) $M(x,y,t)>0$,

(FM-2) $M(x,y,t) = 1$ if and only if $x=y$, (FM-3) M (x,y,t) = M(y,x,t),

(FM-4) M (x,y,t) * M (y,z,s) ≤ M (x,z,t+s),

(FM-5) M (x,y,.) : [0,∞)→[0,1] is continuous.

Let (X, M,*) be a fuzzy metric space. For t>0, the open ball B(x,r,t) with a centre x ϵ X and a radius 0 < r < 1 is defined by

B(x, r, t) ={y ϵ X: M(x,y,t) > 1-r}.

A subset A ⊂ X is called open if for each x ∈ A, there exist t >0 and 0<r<1 such that B (x, r, t) ⊂ A. Let τ denote the family of all open subsets of X.Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 2.1.[18] Let X = R. Denote a*b = a.b for all a,b ϵ [0,1]. For each t ϵ (0, ∞), define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x,y,ϵ X.

Example 2.2.[18] Let X be an arbitrary non-empty set and ψ be an increasing and a continuous function of $\mathbb{R}_+$ into (0, 1) such that $\lim_{t\to\infty} \psi(t) = 1$. Three typical examples of these functions are $\psi(x) = \frac{x}{x+1}$, $\psi(x) = \sin(\frac{\pi x}{2x+1})$ and $\psi(x) = 1-e^{-x}$. Denote a*b = a.b for all a, b ϵ [0, 1]. For each t ϵ (0, ∞), define

$$M(x, y, t) = \psi(t)^{d(x,y)}$$

for all x,y ϵ X, where d(x,y) is an ordinary metric .it is easy to see that (X, M,*) is fuzzy metric space.

Definition 2.3. [15] Let (X, M,*) be a fuzzy metric space

(i) A sequence $\{x_n\}$ in X is said to be convergent to x ϵ X if for each ε >0, and t > 0, there exists n_0ϵ N such that $M(x_n, x, t) > 1-\varepsilon$ for all $n \geq n_0$; i.e., $M(x_n, x, t) \to 1$ as $n\to\infty$ for all t > 0.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy if for each ε>0 and each t>0, there exists n_0ϵN such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$; i.e., $M(x_n, x_m, t) \to 1$ as $n, m \to\infty$ for all t>0.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.1.[9] For all x, y, ϵ X, M (x, y, .) is a non-decreasing function.

Definition 2.4.[18] Let (X, M,*) be a fuzzy metric space. M is said to be continuous on $X^2 \times [0,\infty)$ if $\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$, whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times [0,\infty)$ which converges to a point (x, y, t) ϵ $X^2 \times [0,\infty)$;

i.e. $\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1$ and $\lim_{n\to\infty} M(x, y, t_n) = M(x, y, t)$.

Lemma 2.2. [9] M is a continuous function on $X^2 \times [0, \infty)$.

Definition 2.5.[10] Self mappings A and S of a fuzzy metric space are said to be weakly compatible if they commute at their coincidence points; i.e, Ax=Sx for some x∈X implies that ASx=SAx.

Definition 2.6.[10] Two self maps f and g of a set X are called occasionally weakly compatible iff there is a point x∈ X which is coincidence point of f and g at which f and g commute.

Definition 2.7.[1] The pair (A, S) satisfies the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(x_n, u, t) = 1 \quad \text{for some } x \in X \text{ and all } t > 0.$$

Example 2.3. Let X=R and M $(x,y,t)=\frac{t}{t+|x-y|}$ for every x,y∈ X and t>0. Define A and S by Ax= 2x +1 and Sx = x+2.and the sequence $\{x_n\}$ by $x_n=1+\frac{1}{n}$, n=1, 2…. We have

$$\lim_{n\to\infty} M(Ax_n, 3, t) = \lim_{n\to\infty} M(Sx_n, 3, t) = 1$$

for every t > 0. Then, the pair (A, S) satisfies the property (E.A.). However, A and S are not weakly compatible.

The following example shows that there are some pairs of mappings which do not satisfy the property (E.A.).

Example 2.4. Let X=R and M(x, y, t) = $\frac{t}{t+|x-y|}$ for every x, y, ∈ X and t > 0. Define A and B by Ax= x+1 and Sx =x+2. Assume that there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Sx_n, u, t) = 1$$

for some u ∈ X and all t>0. Therefore

$$\lim_{n\to\infty} M(x_n, u, t) = \lim_{n\to\infty} M(x_n + 2, u, t) = 1$$

We conclude that $x_n \to$u-1 and $x_n \to$u-2 which is a contradiction. Hence, the pair (A, S) does not satisfy property (E.A).

It is our purpose in this paper to prove a common fixed point theorem for occasionally weakly compatible mappings satisfying a contractive condition in fuzzy metric spaces using the property (E.A.).

III. MAIN RESULTS

Let ∅ be the set of all increasing and continuous functions ∅:(0,1]→(0,1], such that ∅(t)>t for every t ∈ (0,1).

Example 3.1. Let ∅ : (0,1]→(0,1] defined by ∅(t)= $t^{1/2}$.

Theorem 3.1. Let (X, M,*) be a fuzzy metric space and S and T be self –mappings of X satisfying the following conditions:

(i) $T(X) \subseteq S(X)$ and T(X) or S(X) is a closed subset of X

(ii) $$M(Tx, Ty, t) \geq \emptyset\left(\min\left\{\begin{array}{c} M(Sx, Sy, t), \\ \sup_{t_1+t_2=\frac{2}{k}t} \min\left\{\begin{array}{c} M(Sx, Tx, t_1), \\ M(Sy, Ty, t_2) \end{array}\right\}, \\ \sup \ldots \left\{M(Sx, Ty, t_{\ldots}) \ldots\right\} \end{array}\right\}\right)$$

for all x, y $\in$ X , t > 0 and for some $1 \leq k < 2$. Suppose that the pair (T, S) satisfies the property (E.A.) and (T, S) is occasionally weakly compatible. Then S and T have a unique common fixed point in X.

Proof. Since the pair (T, S) satisfies the property (E.A.), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Tx_n, z, t) = \lim_{n\to\infty} M(Sx_n, z, t) = 1$$

for some z $\in$ X and every t>0.Suppose that S(X) is a closed subset of X.Then, there exist $v \in X$ such that $Sv = z$ and so

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Sv = z. \qquad (*)$$

Assume that T(X) is a closed subset of X, Therefore, there exist v $\in$ X such that Sv = z. Hence (*) still holds. Now, we show that $Tv = Sv$. suppose that $Tv \neq Sv$. It is not difficult to prove that there exist t_0>0 such that

$$M\left(Tv, Sv, \frac{2}{k}t_0\right) > M(Tv, Sv, t_0) \qquad (**)$$

If not, we have $M(Tv, Sv, t) = M\left(Tv, Sv, \frac{2}{k}t\right)$ for all $t > 0$.

Repeatedly using this equality, we obtains

$$M(Tv, Sv, t) = M\left(Tv, Sv, \frac{2}{k}t\right) = \ldots = M\left(Tv, Sv, \left(\frac{2}{k}\right)^n t\right) \to 1 \quad (n \to \infty).$$

This shows that M (Tv , Sv , t)=1 for all t>0 which contradicts Tv ≠ Sv and so (**) is proved.

Using (ii) we get

$$M(Tx_n, Tv, t_0) \geq \emptyset\left(\min\left\{\begin{array}{c} M(Sx_n, Sv, t_0) \\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min\left\{\begin{array}{c} M(Sx_n, Tx_n, t_1), \\ M(Sv, Tv, t_2) \end{array}\right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max\left\{\begin{array}{c} M(Sx_n, Tv, t_3), \\ M(Sv, Tx_n, t_4) \end{array}\right\} \end{array}\right\}\right)$$

$$\geq \emptyset\left(\min\left\{\begin{array}{c} M(Sx_n, Sv, t_0), \\ \min\left\{M(Sx_n, Tx_n, \varepsilon), M\left(Sv, Tv, \frac{2}{k}t_0 - \varepsilon\right)\right\}, \\ \max\left\{M\left(Sx_n, Tv, \frac{2}{k}t_0 - \varepsilon\right), M(Sv, Tx_n, \varepsilon)\right\} \end{array}\right\}\right)$$

$\forall\, \varepsilon \in \left(0, \frac{2}{k} t_0\right)$. As $n \to \infty$, it follows that

$$M(Sv, Tv, t_0) \geq \emptyset(\min \left\{ \begin{matrix} M(Sv, Sv, t_0), \\ \min \left\{ \begin{matrix} M(Sv, Sv, \varepsilon), \\ M(Sv, Tv, \frac{2}{k} t_0 - \varepsilon) \end{matrix} \right\}, \\ \max \left\{ \begin{matrix} M\left(Sv, Tv, \frac{2}{k} t_0 - \varepsilon\right), \\ M(Sv, Sv, \varepsilon) \end{matrix} \right\} \end{matrix} \right\})$$

$$= \emptyset\left(M\left(Sv, Tv, \frac{2}{k} t_0 - \varepsilon\right)\right)$$

$$> M(Sv, Tv, \frac{2}{k} t_0 - \varepsilon)$$

As $\varepsilon \to 0$, we have

$$M(Sv, Tv, t_0) \geq M(Sv, Tv, \frac{2}{k} t_0)$$

which is a contradiction. Therefore, z=Sv=Tv.Since S and T are occasionally weakly compatible, we have Tz=Sz.

Now,we show that z is a common fixed point of S and T.if $Tz \neq z$

Using (ii) we obtain

$$M(z,Tz,t) \geq \emptyset(\min \left\{ \begin{matrix} M(z, Tz, t), \\ \sup_{t_1 + t_2 = \frac{2}{k} t} \min \left\{ \begin{matrix} M(z, Tz, t_1), \\ M(Sz, Tz, t_2) \end{matrix} \right\}, \\ \sup_{t_3 + t_4 = \frac{2}{k} t} \max \left\{ \begin{matrix} M(z, Tz, t_3), \\ M(Tz, z, t_4) \end{matrix} \right\} \end{matrix} \right\})$$

$$\geq \emptyset(\min \left\{ \begin{matrix} M(z, Tz, t), \\ \min \left\{ \begin{matrix} M\left(z, Tz, \frac{2}{k} t - \varepsilon\right), \\ M(Sz, Tz, \varepsilon) \end{matrix} \right\}, \\ \max \left\{ \begin{matrix} M\left(z, Tz, \frac{2}{k} t - \varepsilon\right), \\ M(Tz, z, \varepsilon) \end{matrix} \right\} \end{matrix} \right\}$$

for all $\varepsilon \in \left(0, \frac{2}{k} t\right)$. as $\varepsilon \to 0$, we have

$$M(z,Tz,t) \geq \emptyset(\min\left\{M(z, Tz, t), M\left(z, Tz, \frac{2}{k} t\right)\right\})$$

$$= \emptyset\big(M(z, Tz, t)\big) > M(z, Tz, t)$$

which is a contradiction. Hence Tz = Sz = z. Thus z is a common fixed point of S and T. The uniqueness of z follows from the inequality (ii).

Example 3.2. Let (X,M,$*$) be a fuzzy metric space,where X=[0,1] with a t-norm defined $a * b = a.b$ for all a,b$\in$[0,1] and ψ is an increasing and a continuous function of R_+ into(0,1) such $\lim_{t\to\infty}\psi(t)$=1.for each t$\in(0,\infty)$, define

$$M(x,y,t)=\psi(t)^{|x-y|}$$

for all x,y$\in$ X, Define self − maps T and S on x as follows:

$$Tx=\frac{x+2}{3},\ Sx = \tan\left(\frac{\pi x}{4}\right)$$

It is easy to see that

(i) $T(X) = [\frac{2}{3}, 1] \subseteq [0,1] = S(X)$,

(ii) for a sequence x_n=1- $\frac{1}{n}$,we have

$$\lim_{n\to\infty} M(Tx_n, 1, t) = \psi(t)^{\left|\frac{1-1/n+2}{3}-1\right|} = 1$$

$$\lim_{n\to\infty} M(Sx_n, 1, t) = \psi(t)^{\left|\tan\left(\frac{\pi(1-1/n)}{4}\right)-1\right|} = 1$$

for every $t > 0$. Hence the pair (T, S) satisfies the property (E.A).it is easy to see that the pair (T,S) is occasionally weakly compatible.Let $\emptyset: (0,1] \to (0,1]$ defined by $\emptyset(t) = t^{1/2}$ as

$$\left|\tan\left(\frac{\pi x}{4}\right) - \tan\left(\frac{\pi y}{4}\right)\right| \geq \frac{\pi}{4}|x-y| .$$

We get

$$M(Tx,Ty,t) = \psi(t)^{\frac{1}{3}|x-y|}$$

$$\geq \psi(t)^{\frac{\pi}{8}|x-y|} = \emptyset(M(Sx, Sy, t)).$$

Thus for $\emptyset(t) = t^{1/2}$ we have

$$M(Tx,Ty,t) \geq \emptyset\left(\min\left\{\begin{matrix} M(Sx, Sy, t), \\ \sup_{t_1+t_2=\frac{2}{k}t} \min\left\{\begin{matrix} M(Sx,Tx,t_1), \\ M(Sy,Ty,t_2) \end{matrix}\right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \max\left\{\begin{matrix} M(Sx,Ty,t_3), \\ M(Sy,Tx,t_4) \end{matrix}\right\} \end{matrix}\right\}\right)$$

for all x,y$\in$ $X, t > 0$ and for some $1 \leq k < 2$. all conditions of theorem 1 hold and z=1 is a unique common fixed point of S and T.

Corollary 3.1. Let T and S be self-mappings of a fuzzy metric space (X, M,*) satisfying the following conditions:

(i) $T^n(X) \subseteq S^m(X)$, $T^n(X)$ or $S^m(X)$ is a closed subset of X and $T^nS=ST^n, TS^m = S^mT$,

(ii) $$M(T^n x, T^n y, t) \geq \emptyset\left(\min\left\{\begin{matrix} M(S^m x, S^m y, t), \\ \sup_{t_1+t_2=\frac{2}{k}t} \min\left\{\begin{matrix} M(S^m x, T^n x, t_1), \\ M(S^m y, T^n y, t_2) \end{matrix}\right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \max\left\{\begin{matrix} M(S^m x, T^n y, t_3), \\ M(S^m y, T^n x, t_4) \end{matrix}\right\} \end{matrix}\right\}\right)$$

for all $x,y \in X$ for some $n, m = 2,3,\ldots, t > 0$ and for some $1 \leq k < 2$.

Suppose that the pair (T^n, S^m) satisfies the property (E.A) and (T^n, S^m) is occasionally weakly compatible. Then S and T have a unique common fixed point in X.

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