MUTUALLY COMPATIBLE MAPPINGS OF TYPE (P) AND FIXED POINT THEOREM IN NON-ARCHIMEDEAN MENGER PM-SPACE

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ABSTRACT

In the present paper, we extend and generalize the result of Cho et. al. [2] by introducing two types of compatible maps in a non-Archimedean Menger PM-space and obtain a common fixed point theorem for six self maps.

Keywords and Phrases -- *Non-Archimedean Menger probabilistic metric space, Common fixed points, Compatible maps, mutually compatible maps*

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I. INTRODUCTION

The notion of non-Archimedean Menger space has been established by Istrătescu and Crivat [4]. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istrătescu [3].

In 1997, Cho et al. [2] introduced the concepts of compatible maps and compatible maps of type (A) in non-Archimedean Menger probabilistic metric spaces and gave some fixed point theorems for these maps. In a paper, Singh, Jain and Jain [10] generalized the result of Cho et. al. [2] by introducing the notion of compatible self maps of type (P-1) and type (P-2).

II. PRELIMINARIES

Definition 1. [2] Let X be a non-empty set and D be the set of all left-continuous distribution functions. An ordered pair (X, F) is called a non-Archimedean probabilistic metric space (shortly a N.A. PM-space) if F is a mapping from X×X into D satisfying the following conditions (the distribution function F(u,v) is denoted by $F_{u,v}$ for all u, v $\in$ X) :

(PM-1) $F_{u,v}(x) = 1$, for all x > 0, if and only if at least two of the three points are equal;

(PM-2) $F_{u,v} = F_{v,u}$;

(PM-3) $F_{u,v}(0) = 0$;

(PM-4) If $F_{u,v}(x_1) = 1$, $F_{v,w}(x_2) = 1$ then $F_{u,w}(\max\{x_1, x_2\}) = 1$

for all $u, v, w \in X$ and $x_1, x_2 \geq 0$.

Definition 2. [2] A t-norm is a function $\Delta : [0,1]\times[0,1]\times[0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each coordinate and $\Delta(a,1,1) = a$ for every $a \in [0,1]$.

Definition 3. [2] A N.A. Menger PM-space is an ordered triple (X, F,Δ), where (X, F) is a non-Archimedean PM-space and Δ is a t-norm satisfying the following condition:

(PM-5) $F_{u,w}(\max\{x,y\}) \geq \Delta(F_{u,v}(x), F_{v,w}(y))$, for all $u, v, w \in X$ and $x, y \geq 0$.

The concept of neighbourhoods in Menger PM-space was introduced by Schweizer and Sklar [8]. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0,1)$, then an (ε,λ) - neighbourhood of x, $U_x(\varepsilon,\lambda)$ is defined by .

$U_x(\varepsilon,\lambda) = \{y \in X : F_{xy}(\varepsilon) > 1 - \lambda\}$.

If the t-norm is continuous and strictly increasing then (X, F, Δ) is a Hausdorff space in the topology induced by the family $U_x(\varepsilon,\lambda) = \{x \in X, \varepsilon > 0, \lambda \in (0,1)\}$ of neighbourhoods.

Definition 4.[2] A N.A. PM-space (X, F, Δ) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that

$$g(F_{x,y}(t)) \leq g(F_{x,z}(t)) + g(F_{z,y}(t))$$

for all $x, y, z \in X$ and $t \geq 0$, where $\Omega = \{g \mid g : [0,1] \to [0, \infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 5. [2] A N.A. Menger PM-space (X, F, Δ) is said to be type $(D)_g$ if there exists a $g \in \Omega$ such that

$$g(\Delta(t_1, t_2)) \leq g(t_1) + g(t_2) \quad \text{for all } t_1, t_2 \in [0,1].$$

Remark 1.

(1) If a N.A. Menger PM-space (X, F, Δ) is of type $(D)_g$ then (X, F, Δ) is of type $(C)_g$.

(2) If a N.A. Menger PM-space (X, F, Δ) is of type $(D)_g$, then it is metrizable, where the metric d on X is defined by

$$d(x,y) = \int_0^1 g\left(F_{x,y}(t)\right) d(t) \quad \text{for all } x, y \in X. \tag{*}$$

Throughout this paper, suppose (X,F,Δ) be a complete N.A. Menger PM-space of type $(D)_g$ with a continuous strictly increasing t-norm Δ.

Let $\phi : [0,+\infty) \to [0, \infty)$ be a function satisfied the condition (Φ) :

(Φ) ϕ is upper-semicontinuous from the right and $\phi(t) < t$ for all $t > 0$.

Lemma 1. [2] If a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ), then we have

(1) For all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, $\phi^n(t)$ is n-th iteration of $\phi(t)$.

(2) If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \ldots$ then $\lim_{n\to\infty} t_n = 0$.

In particular, if $t \leq \phi(t)$ for all $t > 0$, then $t = 0$.

Lemma 2. [2] Let $A, B, S, T : X \to X$ be mappings satisfying the condition (1) and (2) as follows :

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

(2) $g(F_{Ax,By}(t)) \leq \phi(\max\{g(F_{Sx,Ty}(t)), g(F_{Sx,Ax}(t)), g(F_{Ty,By}(T)),$
$\tfrac{1}{2}(g(F_{Sx,By}(T)) + g(F_{Ty,Ax}(t)))\})$

for all t > 0, where a function $\phi : [0,+\infty) \rightarrow [0,+\infty)$ satisfies the condition (Φ). Then the sequence $\{y_n\}$ in X, defined by $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ..., such that

$$\lim_{n\to\infty} g(F_{y_n, y_{n+1}}(t)) = 0 \quad \text{for all } t > 0 \quad \text{is a Cauchy sequence in X.}$$

Definition 6. [10] Let A, S : X $\rightarrow$ X be mappings. A and S are said to be compatible if

$$\lim_{n\to\infty} g(F_{ASx_n,\, SAx_n}(t)) = 0 \qquad \text{for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z in X .

Definition 7. [10] Let A, S : X $\rightarrow$ X be mappings. A and S are said to becompatible maps of type (P) if

$$\lim_{n\to\infty} g(F_{ASx_n,\, SSx_n}(t)) = 0 \text{ and } \lim_{n\to\infty} g(F_{SAx_n,\, AAx_n}(t)) = 0$$

for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z in X.

Definition 8. [10] Let A, S : X $\rightarrow$ X be mappings. A and S are said to be compatible maps of type (P-1) if $\lim_{n\to\infty} g(F_{ASx_n,\, SSx_n}(t)) = 0$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z in X.

Definition 9. [10] Let A, S : X $\rightarrow$ X be mappings. A and S are said to be compatible maps of type (P-2) if $\lim_{n\to\infty} g(F_{SAx_n,\, AAx_n}(t)) = 0$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z in X.

Remark 2. Clearly, if a pair of mappings (A, S) is compatible of type (P-1), then the pair (S, A) is compatible of type (P-2). Such maps are called mutually compatible of type (P) Further, if A and S are compatible mappings of type (P), then the pair (A, S) is compatible of type (P-1) as well as type (P-2). The following is an example of pair of self maps in a N.A. Menger PM-space which are mutually compatible of type (P) but not compatible.

Example 1. Let (X, F, Δ) be the induced N.A. Menger PM-space, where X = [0, 2] and the metric d on X is defined in condition (*) of remark 1. Define self maps A and S as follows :

$$Ax = \begin{cases} 2-x, & \text{if } \; 0 \le x < 1, \\ 2, & \text{if } \; 1 \le x \le 2, \end{cases} \quad \text{and} \quad Sx = \begin{cases} x, & \text{if } \; 0 \le x < 1, \\ 2, & \text{if } \; 1 \le x \le 2. \end{cases}$$

Take $x_n = 1 - 1/n$.

Now $F_{Ax_n, 1}(t) = H(t - (1/n))$

Therefore, $\lim_{n\to\infty} (F_{Ax_n, 1}(t)) = 1$

Then $Ax_n \rightarrow 1$ as $n \rightarrow \infty$ Similarly, $Sx_n \rightarrow 1$ as $n \rightarrow \infty$

Also $F_{ASx_n, SAx_n}(t) = H(t - (1 - 1/n))$ and $F_{ASx_n, SAx_n}(t) = g(H(t-1)) \neq 0, \quad \forall \; t > 0$

Hence, the pair (A,S) is not compatible.

But $F_{ASx_n, SSx_n}(t) = H(t - (2/n))$

And, $\lim_{n\to\infty} g(F_{ASx_n, SSx_n}(t)) = g(H(t)) = 0 \;\; \forall\, t \ge 0.$

Hence, the pair (A,S) is compatible of type (P-1).

Similarly, the pair (A, S) is compatible of type (P-2).

From the above example it is obvious that S and T are mutually compatible but not compatible maps .

Proposition 1. Let B, T : X $\rightarrow$ X be self maps of a N. A. Menger PM-space (X, F, $*$).

(a) If B and T are compatible of type (P-1) and Bz = Tz for some $z \in X$, then BTz = TTz

(b) If B and T are compatible of type (P-2) and Bz = Tz for some $z \in X$, then TBz = BBz

which implies that TBz = BBz = BTz = TTz.

Proof. Suppose $\{x_n\}$ is a sequence in X defined by $x_n = z$, n = 1, 2, ... and Bz = Tz.

Then we have, $Bx_n \to Bz$, $Tx_n \to Tz$ as $n \to \infty$

Since (B, T) is compatible of type (P-1), we have

$$g(F_{BTz,TTz}(t)) = g(F_{BTx_n,TTx_n}(t)) \to 0 \text{ as } n \to \infty$$

i.e. BTz = TTz. (1)

Similarly, we can have

TBz = BBz. (2)

Hence, by (1) and (2), we have BTz = TTz = TBz = BBz.

Proposition 2. Let B and T be self maps of a N. A. Menger PM-space (X, F, $*$) and $\{x_n\}$ is a sequence in X such that Bx_n, $Tx_n \to z$ for some z in X as $n \to \infty$,Then we have the following :

(1) If the pair (B,T) is compatible of type (P-1) then $TTx_n \to Bz$ if B is continuous at z.

(2) If the pair (B,T) is compatible of type (P-2) then $BBx_n \to Tz$ if T is continuous at z

Proof. (i) Suppose T is continuous at z.

Since $Bx_n = Tx_n = z$ for some $z \in X$ and $BTx_n \to Bz$ as $n \to \infty$.

Since (B, T) is compatible of type (P-1), hence we have

$$g(F_{BTx_n, TTx_n}(t)) \to 0 \quad \text{as } n \to \infty.$$

therefore $$g(F_{Bz, TTx_n}(t)) \le g(F_{Bz, BTx_n}(t)) + g(F_{BTx_n, TTx_n}(t)) \to 0 \text{ as } n \to \infty.$$

Hence $TTx_n \to Bz$ as $n \to \infty$ for all $t > 0$

Similarly $BBx_n \to Tz$

III. MAIN RESULT

In the following, we extend the result of Cho et al. [2] to six self maps and generalize it in other respects too.

Theorem 3.1. Let A, B, S, T, L and M be self maps of a complete non-Archimedean Menger PM space (X, F, Δ) satisfying the conditions :

(3.1) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$

(3.2) AB = BA, ST = TS, LB = BL, MT = TM;

(3.3) either AB or L is continuous;

(3.4) (L, AB) and (M, ST) are mutually compatible of type (P)

(3.5) $$g(F_{Lx,My}(t)) \le \phi\,(\max\{g(F_{ABx,STy}(t)), g(F_{ABx, Lx}(t)), g(F_{STy, My}(t)), \tfrac{1}{2}(g(F_{ABx, My}(t)) + g(F_{STy, Lx}(t)))\})$$

for all $t > 0$, where a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ).

Then A, B, S, T, L and M have a unique common fixed point in X.

Proof. Let $x_0 \in X$ □ From condition (3.1)□ $\exists$ $x_1, x_2 \in X$ such that

$$Lx_1 = STx_2 = y_1 \quad \text{and} \quad Mx_0 = ABx_1 = y_0.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(3.6) $\quad Lx_{2n} = STx_{2n+1} = y_{2n} \quad$ and $\quad Mx_{2n+1} = ABx_{2n+2} = y_{2n+1} \quad$ for n = 0, 1, 2, ...

Step 1. We prove that $g(F_{y_n, y_{n+1}}(t)) = 0$ for all $t > 0$.

From (3.5) and (3.6), we have

$$\begin{aligned} g(F_{y_{2n}, y_{2n+1}}(t)) &= g(F_{Lx_{2n}, Mx_{2n+1}}(t)) \\ &\leq \phi(\max\{g(F_{ABx_{2n}, STx_{2n+1}}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), \\ &\qquad g(F_{STx_{2n+1}, Mx_{2n+1}}(t)), \tfrac{1}{2}(g(F_{ABx_{2n}, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, Lx_{2n}}(t)))\}) \\ &= \phi(\max\{g(F_{y_{2n-1}, y_{2n}}(t)), g(F_{y_{2n-1}, y_{2n}}(t)), g(F_{y_{2n}, y_{2n+1}}(t)), \\ &\qquad \tfrac{1}{2}(g(F_{y_{2n-1}, y_{2n+1}}(t)) + g(1))\}) \\ &\leq \phi(\max\{g(F_{y_{2n-1}, y_{2n}}(t)), g(F_{y_{2n}, y_{2n+1}}(t)), \tfrac{1}{2}(g(F_{y_{2n-1}, y_{2n}}(t)) + g(F_{y_{2n}, y_{2n+1}}(t))\}). \end{aligned}$$

If $g(F_{y_{2n-1}, y_{2n}}(t)) \leq g(F_{y_{2n}, y_{2n+1}}(t))$ for all $t > 0$, then by (3.5)

$$g(F_{y_{2n}, y_{2n+1}}(t)) \leq \phi(g(F_{y_{2n}, y_{2n+1}}(t))),$$

On applying Lemma 1, we have

$$g(F_{y_{2n}, y_{2n+1}}(t)) = 0 \quad \text{for all } t > 0.$$

Similarly, we have

$$g(F_{y_{2n+1}, y_{2n+2}}(t)) = 0 \text{ for all } t > 0.$$

Thus, we have $g(F_{y_n, y_{n+1}}(t)) = 0$ for all $t > 0$.

On the other hand, if $g(F_{y_{2n-1}, y_{2n}}(t)) \geq g(F_{y_{2n}, y_{2n+1}}(t))$, then by (3.5), we have

$$g(F_{y_{2n}, y_{2n+1}}(t)) \leq \phi(g(F_{y_{2n-1}, y_{2n}}(t))) \quad \text{for all } t > 0.$$

Similarly, $g(F_{y_{2n+1}, y_{2n+2}}(t)) \leq \phi(g(F_{y_{2n}, y_{2n+1}}(t)))$ for all $t > 0$.

Thus, we have

$$g(F_{y_n, y_{n+1}}(t)) \leq \phi(g(F_{y_{n-1}, y_n}(t))) \quad \text{for all } t > 0 \text{ and } n = 1, 2, 3, \ldots .$$

Therefore, by Lemma 1,

$g(F_{y_n, y_{n+1}}(t)) = 0$ for all $t > 0$, which implies that $\{y_n\}$ is a Cauchy sequence in X by Lemma 2.

Since $(X, \mathbf{F}, \Delta)$ is complete, the sequence $\{y_n\}$ converges to a point $z \in X$. Also its subsequences converges as follows :

(3.7) $\quad \{Mx_{2n+1}\} \to z \quad$ and $\quad \{STx_{2n+1}\} \to z,$

(3.8) $\quad \{Lx_{2n}\} \to z \quad$ and $\quad \{ABx_{2n}\} \to z.$

Case I. AB is continuous.

As AB is continuous, and (L, AB) and (M, ST) are mutually compatible,

$$(AB)^2x_{2n} \to ABz \quad \text{and} \quad (AB)Lx_{2n} \to ABz.$$

As (L, AB) is mutually compatible, so by Proposition 2, we have

$$L(AB)x_{2n} \to ABz.$$

Step 2. Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ for $t > 0$ in (3.5), we get

$$\begin{aligned} g(F_{LABx_{2n},Mx_{2n+1}}(t)) \le \phi\,(\max\{&g(F_{ABABx_{2n},STx_{2n+1}}(t)), g(F_{ABABx_{2n}, LABx_{2n}}(t)),\\ &g(F_{STx_{2n+1}, Mx_{2n+1}}(t)),\\ &\tfrac{1}{2}(g(F_{ABABx_{2n}, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, LABx_{2n}}(t)))\}). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} g(F_{ABz,z}(t)) \le \phi\,(\max\{&g(F_{ABz,z}(t)), g(F_{ABz, ABz}(t)), g(F_{z, z}(t)),\\ &\tfrac{1}{2}(g(F_{ABz, z}(t)) + g(F_{z, ABz}(t)))\}). \end{aligned}$$

i.e. $g(F_{ABz,z}(t)) \le \phi\,(g(F_{ABz,z}(t)))$

which implies that $g(F_{ABz,z}(t)) = 0$ by Lemma 1 and so we have

$$ABz = z.$$

Step 3. Putting $x = z$ and $y = x_{2n+1}$ for $t > 0$ in (3.5), we get

$$\begin{aligned} g(F_{Lz,Mx_{2n+1}}(t)) \le \phi(\max\{&g(F_{ABz,STx_{2n+1}}(t)), g(F_{ABz, Lz}(t)),\\ &g(F_{STx_{2n+1}, Mx_{2n+1}}(t)), \tfrac{1}{2}(g(F_{ABz, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, Lz}(t)))\}). \end{aligned}$$

Letting $n \to \infty$, we get

$$g(F_{Lz,z}(t)) \le \phi(\max\{g(F_{z,z}(t)), g(F_{z, Lz}(t)), g(F_{z, z}(t)), \tfrac{1}{2}(g(F_{z, z}(t)) + g(F_{z, Lz}(t)))\})$$

i.e. $g(F_{Lz,z}(t)) \le \phi(g(F_{Lz,z}(t)))$

which implies that $g(F_{Lz,z}(t)) = 0$ by Lemma 1 and so we have

$$Lz = z.$$

Therefore, $ABz = Lz = z$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ for $t > 0$ in (3.5), we get

$$\begin{aligned} g(F_{LBz,Mx_{2n+1}}(t)) \le \phi(\max\{&g(F_{ABBz,STx_{2n+1}}(t)), g(F_{ABBz, LBz}(t)),\\ &g(F_{STx_{2n+1}, Mx_{2n+1}}(t)), \tfrac{1}{2}(g(F_{ABBz, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, LBz}(t)))\}). \end{aligned}$$

As $BL = LB$, $AB = BA$, so we have

$$L(Bz) = B(Lz) = Bz \quad \text{and} \quad AB(Bz) = B(ABz) = Bz.$$

Letting $n \to \infty$, we get

$$g(F_{Bz, z}(t)) \le \phi(\max\{g(F_{Bz,z}(t)), g(F_{Bz, Bz}(t)), g(F_{z,z}(t)), \tfrac{1}{2}(g(F_{Bz, z}(t)) + g(F_{z, Bz}(t)))\})$$

i.e $g(F_{Bz, z}(t)) \le \phi(g(F_{Bz, z}(t)))$

which implies that $g(F_{Bz, z}(t)) = 0$ and so we have $Bz = z$.

Also, $ABz = z$ and so $Az = z$.

Therefore, $Az = Bz = Lz = z$. (3.9)

Step 5. As $L(X) \subset ST(X)$, there exists $v \in X$ such that $z = Lz = STv$.

Putting $x = x_{2n}$ and $y = v$ for $t > 0$ in (3.5), we get

$$g(F_{Lx_{2n},Mv}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STv}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STv, Mv}(t)), \tfrac{1}{2}(g(F_{ABx_{2n}, Mv}(t)) + g(F_{STv, Lx_{2n}}(t)))\}).$$

Letting $n \to \infty$ and using equation (3.8), we get

$$g(F_{z,Mv}(t)) \leq \phi(\max\{g(F_{z,z}(t)), g(F_{z,z}(t)), g(F_{z,Mv}(t)), \tfrac{1}{2}(g(F_{z,Mv}(t)) + g(F_{z,z}(t)))\})$$

i.e. $g(F_{z,Mv}(t)) \leq \phi(g(F_{z,Mv}(t)))$

which implies that $g(F_{z,Mv}(t)) = 0$ by Lemma 1 and so we have

$$z = Mv.$$

Hence, $STv = z = Mv$.

As (M, ST) is mutually compatible, we have

$$STMv = MSTv.$$

Thus, $STz = Mz$.

Step 6. Putting $x = x_{2n}$, $y = z$ for $t > 0$ in (3.5), we get

$$g(F_{Lx_{2n},Mz}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STz}(t)), g(F_{ABx_{2n}, Lx_{2n},a}(t)), g(F_{STz, Mz}(t)), \tfrac{1}{2}(g(F_{ABx_{2n}, Mz}(t)) + g(F_{STz, Lx_{2n}}(t)))\}).$$

Letting $n \to \infty$ and using equation (3.8) and Step 5, we get

$$g(F_{z,Mz,a}(t)) \leq \phi(\max\{g(F_{z,Mz,a}(t)), g(F_{z,z,a}(t)), g(F_{Mz,Mz,a}(t)), \tfrac{1}{2}(g(F_{z,Mz,a}(t)) + g(F_{Mz,z,a}(t)))\})$$

i.e. $g(F_{z,Mz,a}(t)) \leq \phi(g(F_{z,Mz,a}(t)))$

which implies that $g(F_{z,Mz,a}(t)) = 0$ by Lemma 1 and so we have $z = Mz$.

Step 7. Putting $x = x_{2n}$ and $y = Tz$ for $t > 0$ in (3.5), we get

$$g(F_{Lx_{2n},MTz}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STTz}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STTz, MTz}(t)), \tfrac{1}{2}(g(F_{ABx_{2n}, MTz}(t)) + g(F_{STTz, Lx_{2n}}(t)))\}).$$

As $MT = TM$ and $ST = TS$, we have

$$MTz = TMz = Tz \text{ and } ST(Tz) = T(STz) = Tz.$$

Letting $n \to \infty$ we get

$$g(F_{z,Tz}(t)) \leq \phi(\max\{g(F_{z,Tz}(t)), g(F_{z,z}(t)), g(F_{Tz,Tz}(t)), \tfrac{1}{2}(g(F_{z,Tz}(t)) + g(F_{Tz,z}(t)))\})$$

i.e $g(F_{z,Tz}(t)) \leq \phi(g(F_{z,Tz}(t)))$,

which implies that $g(F_{z,Tz}(t)) = 0$ and so we have $z = Tz$.

Now $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Mz = z$. (3.10)

Combining (3.9) and (3.10), we get

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

Similarly, it is clear that z is also the common fixed point of A, B, S, T, L and M in the case AB is continuous, and (L,AB) and (M,ST) are compatible of type (P-2) .

Case II. L is continuous, and (L, AB) and (M,ST) are mutually compatible.

Since L is continuous, $L^2x_{2n} \to Lz$ and $L(AB)x_{2n} \to Lz$.

As (L, AB) is mutually compatible, so by Proposition 2,

$$(AB)Lx_{2n} \to Lz.$$

Step 8. Putting $x = Lx_{2n}$ and $y = x_{2n+1}$ for $t > 0$ in (3.5), we get

$$\begin{aligned} g(F_{LLx_{2n},Mx_{2n+1}}(t)) \le \ & \phi(\max\{g(F_{ABLx_{2n},STx_{2n+1}}(t)), g(F_{ABLx_{2n}, LLx_{2n}}(t)), \\ & g(F_{STx_{2n+1}, Mx_{2n+1}}(t)), \\ & \tfrac{1}{2}(g(F_{ABLx_{2n}, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, LLx_{2n}}(t)))\}). \end{aligned}$$

Letting $n \to \infty$, we get

$$g(F_{Lz,z}(t)) \le \phi(\max\{g(F_{Lz,z}(t)), g(F_{Lz, Lz}(t)), g(F_{z, z}(t)), \tfrac{1}{2}(g(F_{Lz, z}(t)) + g(F_{z, Lz}(t)))\})$$

i.e. $g(F_{Lz,z}(t)) \le \phi(g(F_{Lz,z}(t)))$,

which implies that $g(F_{Lz,z}(t)) = 0$ and we have $Lz = z$.

Now, using steps 5-7 gives us $Mz = STz = Sz = Tz = z$.

Step 9. As $M(X) \subset AB(X)$, there exists $w \in X$ such that

$$z = Mz = ABw.$$

Putting $x = w$ and $y = x_{2n+1}$ for $t > 0$ in (3.5), we get

$$\begin{aligned} g(F_{Lw,Mx_{2n+1}}(t)) \le \ & \phi(\max\{g(F_{ABw,STx_{2n+1}}(t)), g(F_{ABw, Lw}(t)), \\ & g(F_{STx_{2n+1}, Mx_{2n+1}}(t)), \tfrac{1}{2}(g(F_{ABw, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, Lw}(t)))\}). \end{aligned}$$

Letting $n \to \infty$, we get

$$g(F_{Lw,z}(t)) \le \phi(\max\{g(F_{z, z}(t)), g(F_{z, Lw}(t)), g(F_{z, z}(t)), \tfrac{1}{2}(g(F_{z, z}(t)) + g(F_{z, Lw}(t)))\})$$

i.e. $g(F_{Lw,z}(t)) \le \phi(g(F_{Lw,z}(t)))$,

which implies that $g(F_{Lw,z}(t)) = 0$ and we have

$$Lw = z.$$

Thus, we have $Lw = z = ABw$.

Since (L, AB) is mutually compatible and so by Proposition 1, we have

$LABw = ABLw$ and hence, we have

$$Lz = ABz.$$

Also, $Bz = z$ follows from step 4.

Thus, $Az = Bz = Lz = z$ and we obtain that z is the common fixed point of the six maps in this case also.

Similarly, it is clear that z is also a common fixed point of A, B, S, T, L and M in the case P is continuous, and (L,AB) and (M,ST) are compatible of type (P-2).

Step 10. (Uniqueness) Let u be another common fixed point of A, B, S, T, L and M; then

$$Au = Bu = Su = Tu = Lu = Mu = u.$$

Putting $x = z$ and $y = u$ for $t > 0$ in (3.5), we get

$$g(F_{Lz,Mu}(t)) \leq \phi(\max\{g(F_{ABz,Stu}(t)), g(F_{ABz,Lz}(t)), g(F_{STu,Mu}(t)), \tfrac{1}{2}(g(F_{ABz,Mu}(t)) + g(F_{STu,Lz}(t)))\}).$$

Letting $n \to \infty$, we get

$$g(F_{z,u}(t)) \leq \phi(\max\{g(F_{z,u}(t)), g(F_{z,z}(t)), g(F_{u,u}(t)), \tfrac{1}{2}(g(F_{z,u}(t)) + g(F_{u,z}(t)))\})$$
$$= \phi(g(F_{z,u}(t))),$$

which implies that $g(F_{z,u}(t)) = 0$ and we have $z = u$.

Therefore, z is a unique common fixed point of A, B, S, T, L and M.

This completes the proof.

Remark 3.1. If we take $B = T = I$, the identity map on X in theorem 1, then the condition (3.2) is satisfied trivially and we get

Corollary 3.1. Let A, S, L, M : $X \to X$ be mappings satisfying the condition :

(a) $L(X) \subset S(X)$, $M(X) \subset A(X)$;

(b) Either A or L is continuous;

(c) (L, A) and (M, S) are mutually compatible of type (P)

(d) $$g(F_{Lx,My}(t)) \leq \phi(\max\{g(F_{Ax,Sy}(t)), g(F_{Ax,Lx}(t)), g(F_{Sy,My}(t)), \tfrac{1}{2}(g(F_{Ax,My}(t)) + g(F_{Sy,Lx}(t)))\})$$

for all $t > 0$, where a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ).

Then A, S, L and M have a unique common fixed point in X.

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Cho et. al. [2] in the sense that condition of compatibility of the pairs of self maps in a non-Archimedean Menger PM-space has been restricted to compatible of type (P) in a non-Archimedean Menger PM-space and only one of the mappings of the compatible of type (P-1) or compatible of type(P-2) pair is needed to be continuous.

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