A MATHEMATICAL MODEL FOR STABILITY OF THE FERRO-FLUID MELT IN A SPARSELY PACKED POROUS MEDIUM ROTATING ON A VERTICAL AXIS DURING THE SOLIDIFICATION OF A BINARY ALLOY

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ABSTRACT

In this paper, the problem investigated on the effects of rotation and permeability of a sparsely packed fluid saturated porous medium using the Brinkman model. The study of the cumulative effect of rotation, magnetization, magnetic field concentration and permeability on the behaviour of the solidification problem is considered. The linear analysis and nonlinear analysis are carried out by using a modified power series technique.

Key Words - Binary Alloy, Stability Of Ferro-Fluid Melt, Rotation, Porous Medium

I. INTRODUCTION

The physical configuration constitutes the problem of solutal convection in a horizontal, sparsely packed incompressible and porous layer of ferromagnetic melt in the presence of vertical magnetic field as well as a uniform rotation about the vertical axis and buoyancy forces. From this a semi-infinite slab of crystal is being grown. In this study, the magnetization is a function of concentration. Thus, a concentration gradient is established across the fluid layer. Further, the modified permeability/porous parameter are the additional dimensionless parameter governing the problem. The qualitative and quantitative aspects of the problem are predicted by considering a proper choice of the parameters.

II. MATHEMATICAL FORMULATION

In this study, the mathematical formulation has been constituted through extension of previous studies [3] and [4] as given in below equations:

The Conservation of momentum:

$$\left(\frac{\partial}{\partial t} + \mathbf{q}.\nabla\right)\mathbf{q} + 2\Omega\hat{\mathbf{k}}\times\mathbf{q} = \nabla.(\mathbf{HB}) - \nabla p + \alpha_s gC\,\hat{\mathbf{k}} + \nu_0\frac{\partial\mathbf{q}}{\partial z} + \nu\nabla^2\mathbf{q} - \frac{\nu}{k^*}\mathbf{q} \tag{5.1}$$

The Concentration equation:

$$\left[\rho C_{V,H} - \mu_0\mathbf{H}\cdot\left(\frac{\partial\mathbf{M}}{\partial C}\right)_{V,H}\right]\frac{dC}{dt} + \mu_0 C\left(\frac{\partial\mathbf{M}}{\partial C}\right)_{V,H}\cdot\frac{d\mathbf{H}}{dt} + \rho_c(\mathbf{q}\cdot\nabla)C + \rho_c(\mathbf{q}\cdot\nabla)C_0 = \rho_c\nu_0\frac{\partial C}{\partial z} + \rho_c d\nabla^2 C \tag{5.2}$$

Along these equations, also consider the equations (3.1.3) to (3.1.10) from [3]. In equation (5.1), the viscosity is assumed to be isotropic and independent of the magnetic field where z-axis is vertical.

III. BOUNDARY CONDITIONS

All the boundary conditions i.e. (3.1.11) to (3.1.18) are considered and discussed in [3]. In addition to that the conditions on the vorticity are:

$$\zeta = 0 \text{ at } z = 0; \quad |\zeta| < \infty \text{ as } z \to \infty \tag{5.3}$$

3.1 Basic Solutions

In this study, to get the same basic solutions are discussed in [3] and [6] and the vorticity will be $\boldsymbol{\zeta} = 0$

IV. LINEAR STABILITY ANALYSIS

In this section, the linear stability of the solutal convection in a ferro- porous melt under the influence of magnetization, rotation and magnetic field is being discussed in detail. While the structure of the pattern formation gives the growth of crystal formation. However, the influences of different governing parameters especially rotation on the velocity, concentration and magnetic field profiles are predictable only through nonlinear stability analysis, which is based on the results of the linear stability problem. Therefore, such an investigation is done here to throw light on those aspects of the present solidification problem [3] under rotation.

The perturbations are introduced to study the stability of the quiescent state is discussed in [3] and then (5.1), which can be written in the component form as follows:

$$\left(\frac{\partial}{\partial t}+\mathbf{q}.\nabla\right)u - 2v\,\Omega = -\frac{\partial p}{\partial x}+\mu_0(M_0+H_0)\frac{\partial H_1'}{\partial z}+\nu_0\frac{\partial u}{\partial z}+\nu\nabla^2 u - \frac{\nu}{k_*}u \tag{5.4}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{q}.\nabla\right)v + 2u\,\Omega = -\frac{\partial p}{\partial y}+\mu_0(M_0+H_0)\frac{\partial H_2'}{\partial z}+\nu_0\frac{\partial v}{\partial z}+\nu\nabla^2 v - \frac{\nu}{k_*}v \tag{5.5}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{q}.\nabla\right)w = -\frac{\partial p}{\partial z}+\mu_0(M_0+H_0)\frac{\partial H_3'}{\partial z}-\frac{\mu_0(1-k)\mathbf{K}C_\infty\nu_0}{dk}e^{-\nu_0 z/d}H_3'$$
$$+\frac{\mu_0\mathbf{K}^2C_\infty}{(1+\chi)}\frac{(1-k)\nu_0}{dk}e^{-\nu_0 z/d}C'+\nu_0\frac{\partial w}{\partial z}+\nu\nabla^2 w+\alpha_s g\,C' - \frac{\nu}{k_*}w \tag{5.6}$$

Where, the primes denote the perturbed quantities. Differentiating (5.4), (5.5) and (5.6) w.r.t. x, y, z respectively and adding, we obtain:

$$-2\Omega D\,\zeta = -\frac{\partial}{\partial z}\nabla^2 p+\mu_0(H_0+M_0)D^2(\nabla\cdot\mathbf{H})-\mu^*\left[D^2H_3'-\frac{\nu_0}{d}DH_3'\right]+$$
$$+\mu^*\left[\frac{\nu_0}{d}DH_3'-\frac{\nu_0^2}{d^2}H_3'+\mathbf{K}D^2C-\frac{\mathbf{K}\nu_0}{d}DC-\frac{\mathbf{K}\nu_0}{d}DC+\frac{\mathbf{K}\nu_0^2}{d^2}C\right]+\alpha_s gD^2C',\ \textbf{where}\ D\equiv\frac{\partial}{\partial z} \tag{5.7}$$

Now operate ∇^2 on (5.7) so

that: $\left(\frac{\partial}{\partial t}+\mathbf{q}.\nabla\right)\nabla^2 w = -D\nabla^2 p+\mu_0(H_0+M_0)D\nabla^2H_3'+\mu^*\left[-\nabla^2H_3'-\frac{\nu_0^2}{d^2}H_3'\right]$

$$\mu^*\left[\frac{\mathbf{K}\nu_0^2}{d^2}C'+\mathbf{K}\nabla^2C'\right]+\nu_0\nabla^2 w+\nu\nabla^4 w+\alpha_s g\nabla^2C' - \frac{\nu}{k_*}\nabla^2 w \tag{5.7a}$$

Subtracting (5.7) from the resulting equation (5.7a), to obtain:

$$\left(\frac{\partial}{\partial t}+\mathbf{q}.\nabla\right)\nabla^2 w = \mu^*\left[-D\Delta_2\varphi+\frac{\mathbf{K}}{(1+\chi)}\Delta_2 C-\frac{2\nu_0}{d}D^2\varphi+\frac{2\mathbf{K}\nu_0}{d(1+\chi)}DC\right]+\nu_0 D\nabla^2 w$$
$$+\nu\nabla^4 w+\alpha_s g\Delta_2 C - 2\,\Omega D\zeta - \frac{\nu}{k_*}\nabla^2 w \tag{5.8}$$

Now, the elimination of pressure from (5.4) and (5.5) results in the following vorticity equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{q}.\nabla - \nu_0 D - \nu\nabla^2\right)\zeta = 2\Omega Dw - \frac{\nu}{k_*}\zeta \tag{5.8a}$$

Where, $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the z - component of vorticity, (5.8b)

$$\mu^* = \frac{\mu_0 \mathbf{K}(1-k)\nu_0 C_\infty}{k\, d} e^{-\nu_0 z/d} \quad \text{and} \quad D \equiv \frac{d}{dz}, \ \mathbf{H} = \nabla\varphi, \tag{5.9}$$

Δ_2 is the 2-dimensional Laplacian operator and the primes are dropped for the sake of convenience. Also, the simplification of (5.3) is discussed in [3].

Now, all the three simplified governing equations converted into dimensionless form by using the scales are discussed in [3] and then, the resulting dimensionless set of linearized equations (in the limit of infinite Schmidt number $Sc = \frac{\nu}{d}$) is as follows:

$$\nabla^4 w - M_1 Re^{-z}(\Delta_2 + 2D)H_3 + M_1 Re^{-z}(2D + \Delta_2)C + R\Delta_2 C - N^{1/2} D\zeta - PL\,\nabla^2 w = 0 \tag{5.10}$$

$$\left(D^2 + M_3\Delta_2\right)H_3 - D^2 C = 0 \tag{5.11}$$

$$\left(\nabla^2 + D - \frac{\partial}{\partial t}\right)C + \left(M_2 + M_2^* e^{-z}\right)\frac{\partial H_3}{\partial t} + \left[(1-M_2) - M_2^* e^{-z}\right]e^{-z} w = 0 \tag{5.12}$$

$$N^{1/2} Dw + \nabla^2\zeta - PL\,\zeta = 0 \tag{5.12a}$$

Where, M_1, M_2, M_3 and M_2^* are defined in [3]. Along these, the other dimensionless parameters are defined as follows:

$R = \frac{(1-k)\alpha_s C_\infty d^2}{\nu\nu_0^3 k}$: Solidification Rayleigh number, $N = \frac{4\Omega^2 d^4}{\nu^2\nu_0^4}$: Solidification Taylor number,

$PL = \frac{1}{P_\ell} = \frac{d^2}{\nu_0^2 k_*}$: Modified permeability parameter

The linear stability analysis is carried out by studying the stability of the basic state by the method of infinitesimal perturbation. The approach is similar to that of [3] and [6] and the system was analyzed by using normal modes in the x, y and t variables in the limit of critical wave number approaching zero as k→0. Thus, we set the solution in the following form:

$$(w, C, H_3, \zeta) = \left[F(z), G(z), J(z), F^*(z)\right]e^{(\sigma t + i\hat{k}\cdot r)} \tag{5.13}$$

$$\left.\begin{array}{l} \text{Where, } r = \hat{i}x + \hat{j}y;\ \alpha = \sqrt{\left(k_x^2 + k_y^2\right)} \text{ : the horizontal wave number} \\ \sigma \text{ : The growth rate of the disturbance} \\ \nabla_1^2 f = \Delta_2 f = -\alpha^2 f;\ \nabla^2 f = \left(D^2 - \alpha^2\right)\cdot f;\quad f = f(x,y) \end{array}\right\} \tag{5.14}$$

The substitution of (5.14) with (5.10) to (5.12a) results in the following eigenvalue problem for which σ is the eigenvalue:

$$\left(D^2 - \alpha^2\right)^2 F - M_1 Re^{-z}\left(2D - \alpha^2\right)J - N^{1/2} DF^* + R\left\{M_1 e^{-z}\left(2D - \alpha^2\right) - \alpha^2\right\}G - PL\left(D^2 - \alpha^2\right)F = 0 \tag{5.15}$$

$$N^{1/2} DF + \left(D^2 - \alpha^2\right)\cdot F^* - PLF^* = 0 \tag{5.15a}$$

$$\left(D^2 - M_3\alpha^2\right)J - D^2 G = 0 \tag{5.16}$$

$$\left(D^2 + D - \alpha^2 - \sigma\right)G + \left(M_2 + M_2^* e^{-z}\right)\sigma J + \left\{(1 - M_2) - M_2^* e^{-z}\right\}e^{-z} F = 0 \tag{5.17}$$

With the following boundary conditions:

$$F^* = F = DF = J = (D+1)\,G = 0 \text{ at } z = 0 \tag{5.18}$$

$$|F^*| < \infty, \quad |F| < \infty, \quad |J| < \infty, \quad G = 0 \text{ as } z \to \infty \tag{5.19}$$

In order to obtain the solutions are corresponding to different orders, a power series expansion for the dependent variables including R and σ is assumed as follows:

$$\begin{bmatrix} F \\ G \\ J \\ F^* \\ R \\ \sigma \end{bmatrix} = \begin{bmatrix} F_0 \\ G_0 \\ J_0 \\ F_0^* \\ R_0 \\ \sigma_0 \end{bmatrix} + \alpha^2 \begin{bmatrix} F_1 \\ G_1 \\ J_1 \\ F_1^* \\ R_1 \\ \sigma_1 \end{bmatrix} + \cdots\cdots \tag{5.20}$$

By substituting (5.20) with (5.15) to (5.20) yields the following zeroth– order system of differential equations:

$$\left.\begin{aligned} & D^2\left(D^2 - PL\right)F_0 - 2M_1 R_0 e^{-z}\left(D J_0 - D G_0\right) - N^{1/2} DF_0^* = 0 \\ & N^{1/2} DF_0 + \left(D^2 - PL\right)F_0^* = 0 \\ & D^2 G_0 - D^2 J_0 = 0 \\ & \left(D^2 + D - \sigma_0\right)G_0 + \left(M_2 + M_2^* e^{-z}\right)\sigma_0 J_0 + \left\{(1 - M_2) - M_2^* e^{-z}\right\}e^{-z} F_0 = 0 \end{aligned}\right\} \tag{5.21}$$

The solution of (5.21) is given by:

$F_0 = 0$, $F^* = 0$, $G_0 = g_0 e^{-z}$, $J_0 = g_0 (e^{-z} - 1)$, where g_0 is a constant of integration. (5.22)

The next higher-order system of differential equations, i.e. $\left(O\left(\alpha^2\right)\right)$ are:

$$\left.\begin{aligned} & D^2\left(D^2 - PL\right)F_1 - 2M_1 R_0 e^{-z}\left(D J_1 - D G_1\right) - N^{1/2} DF_1^* = R_0 g_0 (1 + M_1) e^{-z} \\ & N^{1/2} DF_1 + \left(D^2 - PL\right)F_1^* = 0 \\ & D^2 J_1 - M_3 J_0 - D^2 G_1 = 0 \\ & \left(D^2 + D\right)G_1 + \left\{(1 - M_2) - M_2^* e^{-z}\right\} e^{-z} F_1 = \left\{g_0 + \sigma_1 g_0 (1 - M_2 + M_2^*)\right\}e^{-z} - \sigma_1 g_0 M_2^* e^{-2z} + \sigma_1 g_0 M_2 \end{aligned}\right\} \tag{5.23}$$

The solutions of the system (5.23) with the boundary conditions (5.18) and (5.19) are given by:

$$\left.\begin{aligned} & F_1 = f_1 e^{-z} + f_2 z e^{-z} + f_3 e^{-2z} + f_4 \\ & F_1^* = f_1^* e^{-z} + f_2^* z e^{-z} + f_3^* e^{-2z} + f_4^* e^{-z\sqrt{PL}} \\ & G_1 = g_1 z e^{-z} + g_2 e^{-2z} + g_3 z e^{-2z} + g_4 e^{-3z} + g_5 z e^{-3z} + g_6 e^{-4z} \\ & J_1 = j_1 e^{-z} + j_2 z e^{-z} + j_3 e^{-2z} + j_4 z e^{-2z} + j_5 e^{-3z} + j_6 z e^{-3z} + j_7 e^{-4z} + j_8 \end{aligned}\right\} \tag{5.24}$$

Where, $f_1 = -\dfrac{2M_1 M_3 R_0 g_0}{\left\{(1 - PL)^2 + N\right\}} + \dfrac{(4 - PL) M_1 M_3 R_0 g_0}{\left\{(4 - PL)^2 + N\right\}}$, $\quad f_2 = -\dfrac{2M_1 M_3 R_0 g_0 (1 - PL)}{\left\{(1 - PL)^2 + N\right\}}$,

$f_3 = -\dfrac{M_1 M_3 R_0 g_0 (4 - PL)}{2\left\{(4 - PL)^2 + N\right\}}$, $f_4 = \dfrac{2M_1 M_3 R_0 g_0}{\left\{(1 - PL)^2 + N\right\}} + f_3$, $f_1^* = N^{1/2}\left[f_1 + \dfrac{f_2}{(1 - PL)}\left\{\dfrac{2}{(1 - PL)} - 1\right\}\right]$,

$f_2^* = \dfrac{N^{1/2} f_2}{(1 - PL)}$, $\quad f_3^* = \dfrac{2N^{1/2} f_3}{(4 - PL)}$, $\quad f_4^* = -\left(f_1^* + f_3^*\right)$, $a_1 = g_0 + \sigma_1 g_0 \left(1 + M_2^* - M_2\right)$, $a_2 = \sigma_1 g_0 M_2^*$, $a_3 = \sigma_1 g_0 M_2$,

$a_4 = a_1 + f_4(M_2 - 1)$, $a_5 = \{-a_2 + f_1(M_2 - 1) + M_2^* f_4\}$, $a_6 = f_2(M_2 - 1)$, $a_7 = f_3(M_2 - 1) + M_2^* f_1$, $a_8 = M_2^* f_2$, $a_9 = M_2^* f_3$, $g_1 = -a_4$, $g_2 = \frac{1}{2}a_5 + \frac{3}{4}a_6$, $g_3 = \frac{1}{2}a_6$, $g_4 = \frac{1}{6}a_7 + \frac{5}{36}a_8$, $g_5 = \frac{1}{6}a_8$, $g_6 = \frac{1}{12}a_9$, $j_1 = M_3 g_0$, $j_2 = g_1$, $j_3 = g_2$, $j_4 = g_3$, $j_5 = g_4$, $j_6 = g_5$, $j_7 = g_6$, $j_8 = -(j_1 + j_3 + j_5 + j_7)$.

Further, the boundary condition for G_1 i.e., $(D + 1)\,G_1 = 0$ at $z = 0$ (5.25)

Yields the following expression for the growth rate σ_1:

$$\sigma_1 = \frac{\left[\sum_{i=1}^{3} S_i - 1\right]}{S_4} \tag{5.26}$$

Where, $S_1 = M_1 M_3 R_0 \left[\frac{\left\{1 - \frac{(1 - PL)}{2}\right\}}{\left\{(1 - PL)^2 + N\right\}} - \frac{(4 - PL)}{6\left\{(4 - PL)^2 + N\right\}}\right]$, $S_4 = \left(1 - M_2 + M_2^* / 2\right)$

$$S_2 = \frac{M_1 M_3 M_2 R_0}{\left\{(1 - PL)^2 + N\right\}}\left[\frac{(k - 1)/k}{3}\left\{1 - \frac{2}{3}(1 - PL)\right\} - \left\{1 - \frac{(1 - PL)}{2}\right\}\right], \quad S_3 = -\frac{M_1 M_3 M_2 R_0 (4 - PL)}{6\left\{(4 - PL)^2 + N\right\}}\left[\left(\frac{(k - 1)/k}{4}\right) - 1\right]$$

In the limit $M_2 \to 0$, (5.26) reduces to:

$$\sigma_1 = M_1 M_3 R_0 \left[\frac{\left\{1 - \frac{(1 - PL)}{2}\right\}}{\left\{(1 - PL)^2 + N\right\}} - \frac{(4 - PL)}{6\left\{(4 - PL)^2 + N\right\}}\right] - 1 \tag{5.27}$$

In the absence of porous and rotation (5.27) reduces to [3] of (3.3.32).

The condition that σ_1 is real and positive gives:

$$R_0 > \frac{1}{M_1 M_3 (S_5 + S_6 + S_7)} \tag{5.28}$$

Where, $S_6 = \frac{M_1 M_3 M_2}{\left\{(1 - PL)^2 + N\right\}}\left[\frac{(k - 1)/k}{3}\left\{1 - \frac{2}{3}(1 - PL)\right\} - \left\{1 - \frac{(1 - PL)}{2}\right\}\right]$,

$$S_5 = M_1 M_3 \left[\frac{\left\{1 - \frac{(1 - PL)}{2}\right\}}{\left\{(1 - PL)^2 + N\right\}} - \frac{(4 - PL)}{6\left\{(4 - PL)^2 + N\right\}}\right], \quad S_7 = -\frac{M_1 M_3 M_2 (4 - PL)}{6\left\{(4 - PL)^2 + N\right\}}\left[\left(\frac{(k - 1)/k}{4}\right) - 1\right]$$

And

$$R_0 = \frac{1}{M_1 M_3 R_0 \left[\frac{\left\{1 - \frac{(1 - PL)}{2}\right\}}{\left\{(1 - PL)^2 + N\right\}} - \frac{(4 - PL)}{6\left\{(4 - PL)^2 + N\right\}}\right]} \tag{5.29}$$

In the absence of porocity and rotation (5.29) reduces to [3] of (3.3.34).

Thus, for marginal stability: $R_0 = \frac{1}{M_1 M_3 (S_5 + S_6 + S_7)}$ (5.30)

In the next higher approximation, the corresponding differential equations are given by:

$$D^2\left(D^2 - PL\right)F_2 - 2M_1 R_0 e^{-z}(DJ_2 - DG_2) - N^{1/2} DF_2^* = \left(2D^2 - PL\right)F + 2M_1 R_1 e^{-z}(DJ_1 - DG_1)_1 + 2M_1 R_2 e^{-z}(DJ_0 - DG_0) - M_1 R_0 e^{-z}(J_1 - G_1) - M_1 R_1 e^{-z}(J_0 - G_0) + R_1 G_0 + R_0 G_1 \tag{5.31}$$

$$N^{1/2} DF_2 + \left(D^2 - PL\right)F_2^* - F_1^* = 0$$
(5.31a)

$$D^2 J_2 - M_3 J_1 - D^2 G_2 = 0 \tag{5.32}$$

$$\left(D^2 + D\right)G_2 + \left\{(1 - M_2) - M_2^* e^{-z}\right\} e^{-z} F_2 = (1 + \sigma_1)G_1 + \sigma_2 g_0 e^{-z} - \left(M_2 + M_2^* e^{-z}\right)\left\{\sigma_2\left(g_0 e^{-z} - g_0\right) + \sigma_1 j_1\right\} \tag{5.33}$$

The above system of inhomogeneous differential equations is quite complicated along with the complex boundary conditions. While the solution procedure is quite tedious, only the final solutions are presented by avoiding the details, which are as follows:

$$F_2 = f_5 e^{-z} + f_6 z e^{-z} + f_7 e^{-2z} + f_8 z e^{-2z} + f_9 e^{-3z} + f_{10} z e^{-3z} + f_{11} e^{-4z} + f_{12} z e^{-4z} + f_{13} e^{-5z} + f_{14} \tag{5.34}$$

Where, $a_{11} = (2 - PL)f_2 + R_0 g_1 + 2M_1 M_3 R_0 j_8$, $a_{13} = R_0 g_3 - 2M_1 M_3 R_0 j_2$, $a_{15} = R_0 g_5 - M_1 M_3 R_0 j_4$,

$a_{12} = (8 - PL)f_3 + R_0 g_2 - 2M_1 M_3 R_0 j_1 - 2M_1 R_1 j_1 - 2M_1 M_3 R_0 j_2 - M_1 R_0 j_1$, $a_{17} = -\dfrac{2M_1 M_3 R_0 j_6}{3}$,

$a_{14} = R_0 g_4 - M_1 M_3 R_0 j_3 - \dfrac{M_1 M_3 R_0 j_4}{2}$, $a_{16} = R_0 g_6 - \dfrac{2}{3} M_1 M_3 R_0 j_5 - \dfrac{2M_1 M_3 R_0 j_6}{9}$, $a_{18} = -\dfrac{M_1 M_3 R_0 j_7}{2}$,

$b_1 = -(1 - PL)a_{10} + (3 - PL)a_{11} + N^{1/2}\left(f_1^* - 2 f_2^*\right)$, $b_2 = -(1 - PL)a_{11} + N^{1/2} f_2^*$, $b_6 = -(27 - 3PL)a_{15}$,

$b_3 = -(8 - 2PL)a_{12} + (12 - PL)a_{13} + 4N^{1/2} f_3^*$, $b_4 = -(8 - 2PL)a_{13}$, $b_5 = -(27 - 3PL)a_{14} + (27 - PL)a_{15}$,

$b_7 = -(64 - 4PL)a_{16} + (48 - PL)a_{17}$, $b_8 = -(64 - 4PL)a_{17}$, $b_9 = -(125 - 5PL)a_{18}$, $b_{10} = \left\{(1 - PL)^2 + N\right\}$

, $b_{11} = \left\{(4 - PL)^2 + N\right\}$, $b_{12} = \left\{(9 - PL)^2 + N\right\}$, $b_{13} = \left\{(16 - PL)^2 + N\right\}$, $b_{14} = \left\{(25 - PL)^2 + N\right\}$,

$$f_5 = -\frac{b_2}{b_{10}} + \frac{0.25}{b_{11}}\left[(b_3 + b_4) + \frac{8(4 - PL)\cdot b_4}{b_{11}}\right] + \frac{1}{27 b_{12}}\left\{\left(b_5 + \frac{2b_6}{3}\right) + \frac{12(9 - PL) b_6}{b_{12}}\right\} + \frac{1}{16 b_{13}}\left\{\left(b_7 + \frac{b_8}{2}\right) + \frac{16(16 - PL) b_8}{b_{13}}\right\} + \frac{b_9}{25 b_{14}}$$

$f_6 = -\dfrac{b_2}{b_{10}}$, $f_7 = -\dfrac{1}{8b_{11}}\left[\left(b_3 + \dfrac{3}{2} b_4\right) + \dfrac{8(4 - PL) b_4}{b_{11}}\right]$, $f_8 = -\dfrac{b_4}{8b_{11}}$, $f_9 = -\dfrac{1}{27b_{12}}\left[(b_5 + b_6) + \dfrac{12(9 - PL) b_6}{b_{12}}\right]$,

$f_{10} = -\dfrac{b_6}{27 b_{12}}$, $f_{11} = -\dfrac{1}{64b_{13}}\left[\left(b_7 + \dfrac{3b_8}{4}\right) + \dfrac{16(16 - PL) b_8}{b_{13}}\right]$, $f_{12} = -\dfrac{b_8}{64 b_{13}}$, $f_{13} = -\dfrac{b_9}{125 b_{14}}$

$f_{14} = -f_6 + f_7 - f_8 + 2 f_9 - f_{10} + 3 f_{11} - f_{12} + 4 f_{13}$,

Further,

$$F_2^* = f_5^* e^{-z} + f_6^* z e^{-z} + f_7^* e^{-2z} + f_8^* z e^{-2z} + f_9^* e^{-3z} + f_{10}^* z e^{-3z} + f_{11}^* e^{-4z} + f_{12}^* z e^{-4z} + f_{13}^* z e^{-5z} + f_{14}^* \tag{5.34a}$$

Where, $f_5^* = \left[f_1^* + N^{1/2}(f_5 - f_6) + \dfrac{2\left(N^{1/2} f_6 + f_2^*\right)}{(1 - PL)}\right]\dfrac{1}{(1 - PL)}$, $f_6^* = \dfrac{f_2^* + N^{1/2} f_6}{(1 - PL)}$, $f_8^* = \dfrac{2N^{1/2} f_8}{(4 - PL)}$,

$f_7^* = \left[f_3^* + N^{1/2}(2f_7 - f_8) + \dfrac{8N^{1/2} f_8}{(4 - PL)}\right]\dfrac{1}{(4 - PL)}$, $f_9^* = \left[(3f_9 - f_{10})N^{1/2} + \dfrac{18N^{1/2} f_{10}}{(9 - PL)}\right]\dfrac{1}{(9 - PL)}$, $f_{10}^* = \dfrac{3N^{1/2} f_{10}}{(9 - PL)}$,

$f_{11}^* = \left[N^{1/2}(4 f_{11} - f_{12}) + \dfrac{32 N^{1/2} f_{12}}{(16 - PL)}\right]\dfrac{1}{(16 - PL)}$, $f_{12}^* = \dfrac{4N^{1/2} f_{12}}{(16 - PL)}$, $f_{13}^* = \dfrac{5N^{1/2} f_{13}}{(25 - PL)}$,

$f_{14}^* = -\left(f_5^* + f_7^* + f_9^* + f_{11}^* + f_{13}^*\right)$

$$G_2 = g_7 z e^{-z} + g_8 z^2 e^{-z} + g_9 e^{-2z} + g_{10} z e^{-2z} + g_{11} e^{-3z} + g_{12} z e^{-3z} + g_{13} e^{-4z} + g_{14} z e^{-4z} + g_{15} e^{-5z} + g_{16} z e^{-5z} + g_{17} e^{-6z} + g_{18} z e^{-6z} + g_{19} e^{-7z} \tag{5.35}$$

Where, $a_{19} = \sigma_2 g_0\left(1+M_2^* - M_2\right) - \sigma_1\left(M_2^* j_8 + M_2 j_1\right) - (1-M_2) f_{14}$, $a_{20} = (1+\sigma_1) g_1 - M_2 \sigma_1 j_2$,

$a_{21} = (1+\sigma_1) g_2 - M_2^* \sigma_2 g_0 - \sigma_1\left(M_2 j_3 + M_2^* j_1\right) - (1-M_2) f_5 + M_2^* f_{14}$, $a_{26} = -M_2^* \sigma_1 j_6 - (1-M_2) f_{10} + M_2^* f_8$

$a_{22} = (1+\sigma_1) g_3 - M_2^* \sigma_1 j_2 - \sigma_1 M_2 j_4 - (1-M_2) f_6$, $a_{23} = (1+\sigma_1) g_4 - M_2^* \sigma_1 j_3 - \sigma_1 M_2 j_5 - (1-M_2) f_7 + M_2^* f_5$,

$a_{24} = (1+\sigma_1) g_5 - M_2^* \sigma_1 j_4 - \sigma_1 M_2 j_6 - (1-M_2) f_8 + M_2^* f_6$, $a_{27} = -M_2^* \sigma_1 j_7 - (1-M_2) f_{11} + M_2^* f_9$,

$a_{25} = (1+\sigma_1) g_6 - M_2^* \sigma_1 j_5 - \sigma_1 M_2 j_7 - (1-M_2) f_9 + M_2^* f_7$, $a_{28} = -(1-M_2) f_{12} + M_2^* f_{10}$, $a_{30} = M_2^* f_{12}$,

$a_{29} = -(1-M_2) f_{13} + M_2^* f_{11}$, $a_{31} = M_2^* f_{13}$, $b_{19} = \sigma_1\left(M_2 j_1 + M_2^* j_8\right) + (1-M_2) f_{14}$,

$b_{21} = \frac{1}{2}\left\{\sigma_1\left(M_2 j_3 + M_2^* j_1\right) + (1-M_2) f_5 - (1+\sigma_1) g_2 - M_2^* f_{14}\right\}$, $g_7 = -(a_{19} + a_{20})$, $g_8 = -\frac{1}{2} a_{20}$, $g_{10} = \frac{a_{22}}{2}$,

$g_9 = \frac{a_{21}}{2} + \frac{3a_{22}}{4}$, $g_{11} = \frac{a_{23}}{6} + \frac{5a_{24}}{36}$, $g_{12} = \frac{a_{24}}{6}$, $g_{13} = \frac{a_{25}}{12} + \frac{7a_{26}}{144}$, $g_{14} = \frac{a_{26}}{12}$, $g_{15} = \frac{a_{27}}{20} + \frac{9a_{28}}{400}$, $g_{16} = \frac{a_{28}}{20}$,

$g_{17} = \frac{a_{29}}{30} + \frac{11a_{30}}{900}$, $g_{18} = \frac{a_{30}}{30}$, $g_{19} = \frac{a_{31}}{42}$

And

$$J_2 = j_9 e^{-z} + j_{10} z e^{-z} + j_{11} z^2 e^{-z} + j_{12} e^{-2z} + j_{13} z e^{-2z} + j_{14} e^{-3z} + j_{15} z e^{-3z} + j_{16} e^{-4z} + j_{17} z e^{-4z} + j_{18} e^{-5z} + j_{19} z e^{-5z} + j_{20} e^{-6z} + j_{21} z e^{-6z} + j_{22} e^{-7z} + j_{23} \tag{5.36}$$

Where, $j_9 = M_3(j_1 + 2j_2)$, $j_{10} = M_3 j_2 + g_7$, $j_{11} = g_8$, $j_{12} = \frac{M_3 j_3}{4} + \frac{M_3 j_4}{4} + g_9$

, $j_{13} = \frac{M_3 j_4}{4} + g_{10}$, $j_{14} = M_3\left(\frac{j_5}{9} + \frac{2j_6}{27}\right) + g_{11}$, $j_{15} = \frac{M_3 j_6}{9} + g_{12}$, $j_{16} = \frac{M_3 j_7}{16} + g_{13}$, $j_{17} = g_{14}$, $j_{18} = g_{15}$,

$j_{19} = g_{16}$, $j_{20} = g_{17}$, $j_{21} = g_{18}$, $j_{22} = g_{19}$, $j_{23} = -[j_9 + j_{12} + j_{14} + j_{16} + j_{18} + j_{20} + j_{22}]$

Finally the conditions:

$(D+1) G_2 = 0$ at $z = 0$ (5.37)

Gives the expression for the second – order growth rate σ_2 :

$$\sigma_2 = \frac{1}{P_5} \sum_{i=1}^{4} P_i \tag{5.38}$$

Where, $P_1 = b_{19} - a_{20} + b_{21}$, $P_2 = -\left(\frac{a_{22}}{4} + \frac{a_{23}}{3} + \frac{a_{24}}{9}\right)$, $P_3 = -\left(\frac{a_{25}}{4} + \frac{a_{26}}{16} + \frac{a_{27}}{5}\right)$,

$$P_4 = -\left(\frac{a_{28}}{25} + \frac{a_{29}}{6} + \frac{a_{30}}{36} + \frac{a_{31}}{7}\right), \quad P_5 = \left(1 - M_2 + \frac{M_2^*}{2}\right) g_0$$

The results are presented through a number of graphs.

V. NONLINEAR ANALYSIS

The nonlinear analysis of any physical problem is capable of predicting several qualitative as well as the quantitative aspects of the problem under consideration. Therefore, in this section, the weakly nonlinear behaviour of the system consisting of the ferromagnetic melt with solidification front at the interface subjected to uniform rotation and magnetic field are investigated. Now, express the variables q, B and H in terms of the poloidal components as:

$$(q, B, H, \zeta) = \delta\,(\varphi, \psi, E, \varphi^*) \tag{5.39}$$

With, $\delta = \nabla \times \nabla \times \hat{\mathbf{k}}$ (5.40)

Substituting these polodial components into the system (3.1.3) to (3.1.8) in [3] and also, in the system (5.1) to (5.2). Then, to get the following nonlinear system of equations after some mathematical simplification**:**

$$\Delta_2\left[\left(D^2 + \Delta_2\right)\left(D^2 + \Delta_2 - PL\right)\varphi - RC\right] + N^{1/2} \Delta_2 D \varphi^* = -M_1 R \left\{\delta\,(\delta\psi \cdot \nabla \delta E) \cdot \hat{\mathbf{k}}\right\} \tag{5.41}$$

$$\Delta_2\left[N^{1/2}D\varphi+\left(\Delta_2+D^2-PL\right)\varphi^*\right]=0 \quad (5.41a)$$

$$\Delta_2\left(D^2+M_3\Delta_2\right)E+D^2C=0 \quad (5.42)$$

$$\left(\nabla^2+D-\frac{\partial}{\partial t}\right)C-\Delta_2\left\{(1-M_2)-M_2^*e^{-z}\right\}e^{-z}\varphi-\Delta_2\left(M_2+M_2^*e^{-z}\right)\frac{\partial E}{\partial t}=\delta\varphi\cdot\nabla C \quad (5.43)$$

The analysis is carried out for small k also. The governing system of nonlinear differential equations in terms of the rescaled variables (rescaled variables are Ref [4]) takes the form (after dropping the primes):

$$\Delta_2\left\{\left(\varepsilon\Delta_2+D^2\right)\left(\varepsilon\Delta_2+D^2-PL\right)\varphi-\varepsilon R_0(1+\varepsilon\gamma)C+N^{1/2}D\varphi^*\right\}=-M_1\varepsilon R_0(1+\varepsilon\gamma)\left\{\delta\left(\delta\psi\cdot\nabla\delta E\right)\cdot\hat{\mathbf{k}}\right\} \quad (5.44)$$

$$\Delta_2\left[N^{1/2}D\varphi+\left(\varepsilon\Delta_2+D^2-PL\right)\varphi^*\right]=0 \quad (5.44a)$$

$$\Delta_2\left(D^2+\varepsilon M_3\Delta_2\right)E+D^2C=0 \quad (5.45)$$

$$\left(D^2+D+\varepsilon\Delta_2-\varepsilon^2\frac{\partial}{\partial t}\right)C-\Delta_2\left\{(1-M_2)-M_2^*e^{-z}\right\}e^{-z}\varphi-\varepsilon^2\Delta_2\left(M_2+M_2^*e^{-z}\right)\frac{\partial E}{\partial t}=\delta\varphi\cdot\nabla C \quad (5.46)$$

Together with:

$$\left.\begin{array}{ll}\varphi^*=\varphi=D\varphi=E=\left(D+1-\varepsilon^2 k_1\right)C=0 & \text{at} \quad z=0\\ \left|\varphi^*\right|<\infty,\left|\varphi\right|<\infty,\quad \left|E\right|<\infty,\quad C=0 & \text{as} \quad z\to\infty\end{array}\right\} \quad (5.47)$$

The solution of the nonlinear system (5.41) to (5.46) subject to the boundary conditions (5.47) in terms of a power series in ε, of the form:

$$\begin{bmatrix}\varphi\\ E\\ C\\ \psi\\ \varphi^*\end{bmatrix}=\sum_{m=1}\varepsilon^m\begin{pmatrix}\varphi_m\\ E_m\\ C_m\\ \psi_m\\ \varphi_m^*\end{pmatrix}$$

(5.48)

Substituting (5.48) into (5.44) to (5.47) and the corresponding differential equations of O (ε) are as follows:

$$\left.\begin{array}{l}\Delta_2D^2\left(D^2-PL\right)\varphi_1+N^{1/2}\Delta_2D\,\varphi_1^*=0\\ \Delta_2\left[N^{1/2}D\varphi_1+\left(D^2-PL\right)\varphi_1^*\right]=0\\ \Delta_2D^2E_1+D^2C_1=0\\ \left(D^2+D\right)C_1-\Delta_2\left\{(1-M_2)-M_2^*e^{-z}\right\}e^{-z}\varphi_1=0\end{array}\right\} \quad (5.49)$$

Together with:

$$\left.\begin{array}{l}\varphi_1=\varphi_1^*=D\varphi_1=E_1=(D+1)C_1=0 \quad \text{at} \quad z=0\\ \left|\varphi_1^*\right|<\infty,\left|\varphi_1\right|<\infty,\quad \left|E_1\right|<\infty,\quad C_1=0 \text{ as } \quad z\to\infty\end{array}\right\} \quad (5.50)$$

On solving the above system, to obtain:

$$\varphi_1^*=0;\quad \varphi_1=0;\quad C_1=Ae^{-z},\quad E_1=\frac{A}{\Delta_2}\left(1-e^{-z}\right) \quad (5.51)$$

Where, A = A (x, y, t) is an amplitude function. (5.52)

In order to obtain more accurate results, the higher order solutions are computed i.e. $\left(O\left(\varepsilon^2\right)\right)$:

$$D^2\left(D^2-PL\right)\varphi_2+N^{1/2}D\varphi_2^*=R_0Ae^{-z} \quad (5.53)$$

$$N^{1/2}D\varphi_2+\left(D^2-PL\right)\varphi_2^*=0 \quad (5.54)$$

$$\Delta_2D^2E_2+D^2C_2=M_3A\Delta_2e^{-z}-M_3\Delta_2A=M_3\Delta_2A\left(e^{-z}-1\right) \quad (5.55)$$

$$\left(D^2 + D\right)C_2 - \Delta_2\left\{(1 - M_2) - M_2^* e^{-z}\right\} e^{-z}\varphi_2 = -\Delta_2 A e^{-z} \tag{5.56}$$

By using the corresponding boundary conditions, the solution of the above system are given by:

$$\varphi_2^* = \frac{F_1}{Đ} \tag{5.57}$$

Where, $Đ = -\left(\frac{M_1 M_3 g_0}{A}\right)\left[\frac{(1 - PL)}{b_{10}} - 2 + \frac{(4 - PL)b_{10}}{b_{11}} - 4PL\right]$

Equation (5.58) can be written as:

$$\varphi_2^* = q_1^* e^{-z} + q_2^* z e^{-z} + q_3^* e^{-2z} + q_4^* \tag{5.59}$$

Where, $q_1^* = \frac{f_1}{Đ}, q_2^* = \frac{f_2}{Đ}, q_3^* = \frac{f_3}{Đ}, q_4^* = \frac{f_4}{Đ}$

Correspondingly, we can write

$$\varphi_2 = q_1 e^{-z} + q_2 z e^{-z} + q_3 e^{-2z} + q_4 z + q_5 \tag{5.60}$$

Where, $q_1 = N^{-1/2}\left\{(1 - PL)q_1^* - (1 + PL)q_2^*\right\},\quad q_2 = N^{-1/2}(1 - PL)q_2^*,\quad q_3 = 2N^{-1/2}\left(2 - \frac{PL}{2}\right)q_3^*,$

$q_4 = N^{-1/2} PLq_4^*,\quad q_5 = -(q_1 + q_3)$

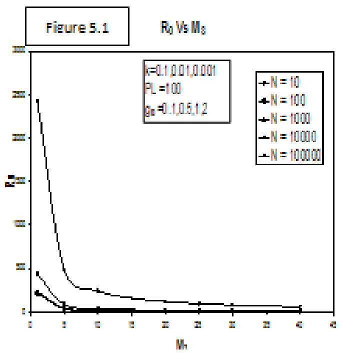
$$C_2 = -\frac{\Delta_2 A}{g_0} G_1 \tag{5.61}$$

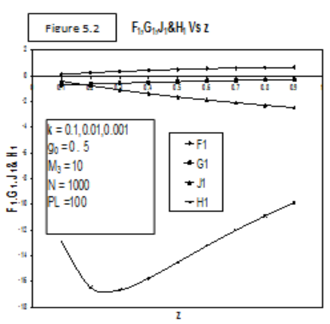
$$E_2 = \frac{A}{g_0} J_1 \tag{5.62}$$

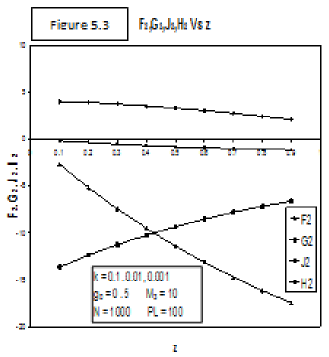
Where, the expressions for F_1, G_1 and J_1 are given by (5.24). The evolution equation can be obtained by integrating (5.46), in the order of ε^3, with respect to z from z = 0 to ∞ and using (5.57) and the solutions corresponding to O (ε^3). As the equations are highly complicated, the expressions are not presented here for the determination of the amplitude A.

VI. RESULTS AND DISCUSSIONS

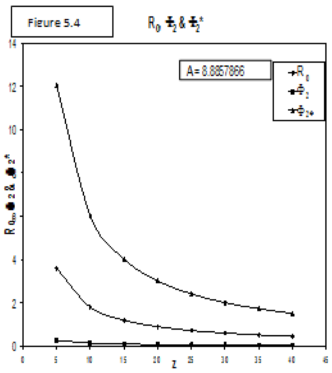
The results are presented in the form of graphs for a wide range of parameters. The graphs corresponding to linear theories as well as nonlinear theories are discussed in detail. The graphs reveal the following points: In figure 5.1, the graph of R_0 vs. M_3 are presented for different values of N. It was found that R_0 decreases tremendously with M_3 for all values of N. Also, the figures 5.2 and 5.3 present a comparison of the profiles for F_1, G_1, J_1 and H_1 as well as a comparison of the profiles F_2, G_2, J_2 and H_2 for k=0.001, g_0 = 0.5, M_3 = 10, N = 10^3 and PL = 10^2.

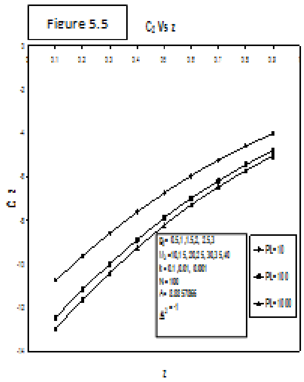


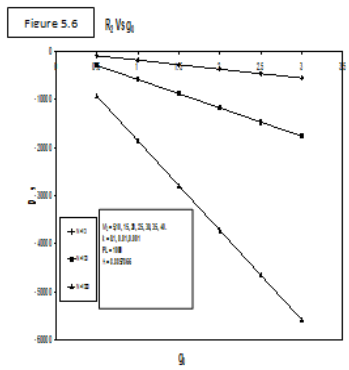




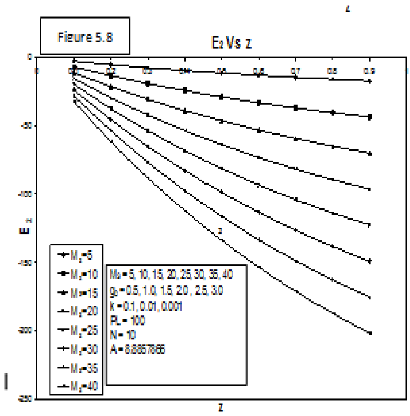
In figure 5.4, the curves for R_0, ϕ_2 and ϕ_2^* are presented for A = 8.8857866. All the three curves gradually decrease with z and the values are high in the range $5 \le z \le 10$. For any specific value of z, it was found that $F_2 < R_0 < F_2^*$. Also in figure 5.5, the effect of increase in PL is to reduce the value of C_2) for fixed values of the other parameters. While, the figure 5.6 represents the graph of R_2 vs. g_0 is presented for fixed values of z, PL and A. As per the physical situation, R_0 has very significant value for rotation rates N = 10, 10^2, 10^3 and for large values of g_0. The interesting feature observed is that the Ferro-convective instability is **effective** only in the **large rotation rates** (N > 10^3) irrespective of the values of g_0

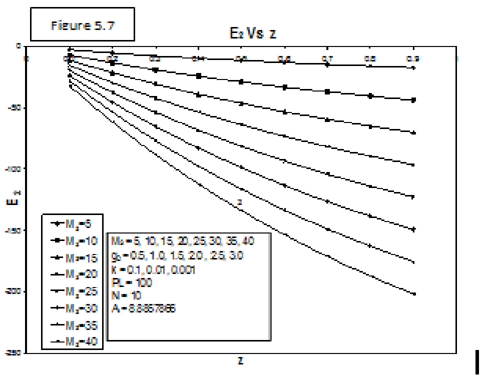






The graphs of E_2 vs. z are presented in the figures 5.7 and 5.8 in order to study the variation of the rotation and morphological parameters of E_2. It was observed that increase in N causes an enhancement in the values of E_2 and increase in M_3 causes reduction in the values of E_2.





VII. CONCLUSION

Finally, it is concluded that, these graphs are of immense use in predicting the influences of the different parameters either individually or cumulatively on the functions considered in a clear manner. Further, this study predicts the nature of the solidification in a sparsely packed porous medium rotating on a vertical axis.

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