FIXED POINT THEOREMS IN INTUITIONISTIC FUZZY METRIC SPACE

Arihant Jain[1], Bijendra Singh[2], Mamta Singh[3], Amit Kumar Govery[4]

[1]*Department of Applied Mathematics,*

Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P.)

[2]*School of Studies in Mathematics, Vikram University, Ujjain (M.P.)*

[3]*Department of Mathematical Science and Computer Application,*

Bundelkhand University, Jhansi (U.P.)

[4]*Department of Mathematics, Mandsaur Institute of Technology, Mandsaur (M.P.)*

ABSTRACT

In this paper, common fixed point theorems for occasionally weakly compatible mappings in intuitionistic fuzzy metric space has been proved which is a generalization of the result of Turkoglu et. al. [9]. We also cited an example in support of our result.

Keywords - Common Fixed Points, Intuitionistic Fuzzy Metric Space, Compatible Maps And Occasionally Weakly Compatible Mappings

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I. INTRODUCTION

Zadeh's [10] investigation of the notion of fuzzy set has led to a rich growth of fuzzy Mathematics. Many authors have introduced the concept of fuzzy metric in different ways. Atanassov [2] introduced and studied the concept of intuitionistic fuzzy set. The notion of intuitionstic fuzzy metric space defined by Park [8] is a generalization of fuzzy metric space due to George and Veeramani [4]. Further, using the idea of Intuitionistic Fuzzy set, Alaca et.al. [1] defined the notion of Intuitionistic Fuzzy Metric space, as Park [8] with the help of continuous t – norms and continuous t–conorms, as a generalization of fuzzy metric space due to Kramosil and Michalek [6], further Coker [3], Turkoglu [9] and others have been expansively developed the theory of Intuitionistic Fuzzy set and applications. Turkoglu et. al. [9] introduced the notion of Cauchy sequences in intuitionistic fuzzy metric space. They generalized the Jungck's [5] common fixed point theorem in intuitionistic fuzzy metric space and proved the intuitionistic fuzzy version of Pant's theorem [7] by giving the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric space.

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

II. PRELIMINARIES

Definition 2.1. [1] A binary operation $*:[0,1] \times [0, 1] \rightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions :

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c, d $\in [0,1]$.

Definition 2.2. [1] A binary operation $\Diamond: [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-conorm if satisfies the following conditions :

(i) $\Diamond$ is commutative and associative;

(ii) $\Diamond$ is continuous;

(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$;

(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$

for all a, b, c, d $\in [0,1]$.

Remark 2.1. [1] The concepts of triangular norms (t-norms) and triangular co-norms (t-conorms) are known as axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively.

Definition 2.3. [1] A 5-tuple (X,M,N,*,) is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, is a continuous t-conorm and M, N are fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions :

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \rightarrow [0,1]$ is left continuous;

(vii) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, $N(x, y, .) : [0,\infty) \to [0,1]$ is right continuous ;

(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$ for all x, y in X.

Then (M, N) is called an *intuitionistic fuzzy metric* on X. The functions M(x,y,t) and N(x,y,t) denote the degree of nearness and the degree of non nearness between x and y with respect to t, respectively.

Remark 2.1. [1] Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form (X, M, 1-M, *, ◊) such that t-norm * and t-conorm are associated, i.e,

$$x \lozenge y = 1- (1 - x) * (1 - y) \quad \text{for all } x, y \in X.$$

Remark 2.2. [1] In intuitionistic fuzzy metric space X, M(x, y, .) is non-decreasing and N(x, y, .) is non-increasing for all $x, y \in X$.

Definition 2.4. [1] Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be *Cauchy sequence* if, for all $t > 0$ and $p > 0$,

$$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.$$

(b) a sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n \to \infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_n, x, t) = 0$$

Definition 2.5. [1] An intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be *complete* if and only if every Cauchy sequence in X is convergent.

Lemma 2.1. [2] Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. If there exists $k \in (0,1)$ such that

$$M(x,y,kt) \geq M(x,y,t) \text{ and } N(x,y,kt) \leq N(x,y,t) \text{ for } x, y \in X.$$

Then x = y.

Lemma 2.2. [2] Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. If there exists a number $k \in (0,1)$ such that

(a) $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, $\quad N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$

for all $t > 0$ and n = 1, 2, ... then $\{y_n\}$ is a Cauchy sequence in X.

Definition 2.6.[1] Two maps A and B from an intuitionistic fuzzy metric space (X, M, N, *, ◊) into itself are said to *compatible* if

$$\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \quad \text{for some } x \in X.$$

Definition 2.7. Two maps A and B from an intuitionistic fuzzy metric space (X, M, N, *, ◊) into itself are said to *occasionally weakly compatible* if and only if there is a point x in X which is coincidence point of A and B at which A and B commute.

III. MAIN RESULT

In this section, we prove a common fixed point theorem for occasionally weakly compatible mappings in intuitionistic fuzzy metric space which is a generalization of Turkoglu et. al. [9].

Theorem 3.1. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric spaces. Let f, g : $X \to X$ be occasionally weakly compatible mappings satisfying the following conditions :

(3.1) $g(X) \subset f(X)$;

(3.2) there exists a number $k \in (0, 1)$ such that

$$M(gx, gy, kt) \geq M(fx, fy, t)$$

$$N(gx, gy, kt) \leq N(fx, fy, t)$$

for all $x, y \in X$ and $t > 0$;

(3.3) one of the subspace g(X) or f(X) is complete.

Then f and g have a unique common fixed point in X.

Proof. By (3.1), since $g(X) \subset f(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $gx_0 = fx_1$.

In general, chose x_{n+1} such that

$$y_n = fx_{n+1} = gx_n.$$

From Turkoglu et. al. [9], we conclude that $\{y_n\}$ is a Cauchy sequence in X.

Since either f(X) or g(X) is complete, for definiteness assume that f(X) is complete.

Since f(X) is complete, so there exists a point $p \in X$ such that

$$fp = z.$$

Now using (3.2), we have

$$M(gp, gx_n, kt) \geq M(fp, fx_n, t)$$

and $$N(gp, gx_n, kt) \leq N(fp, fx_n, t).$$

Taking limit as $n \to \infty$, we obtain

$$gp = z.$$

Therefore, we have fp = gp =z.

Since f and g are occasionally weakly compatible, therefore

$$fgp = gfp, \text{ i.e., } fz = gz.$$

Now we show that z is a common fixed point of f and g.

From (3.2), we get

$$M(gz, gx_n, kt) \geq M(fz, fx_n, t)$$

and $$N(gz, gx_n, kt) \leq N(fz, fx_n, t)$$

Proceeding limit as $n \to \infty$, we obtain

$$gz = z.$$

Hence z is a common fixed point of f and g both.

Uniqueness : Let w be another common fixed point of f and g then

$$fw = gw = w.$$

Using (3.2), we get

$$M(gz, gw, kt) \geq M(fz, fw, t)$$

$$N(gz, gw, kt) \leq N(fz, fw, t)$$

or $$M(z, w, kt) \geq M(z, w, t)$$

$$N(z, w, kt) \leq N(z, w, t).$$

Using lemma 2.1, we get

$$z = w.$$

Hence, z is the unique common fixed point of f and g.

This completes the proof.

Example 3.1. Let $X = \left\{ \frac{1}{n} : n \in N \right\} \cup \{0\}$ with * continuous t-norm and ◊ continuous t-conorm defined by a * b = ab and a ◊ b = min {1, a+ b} respectively, for all a, b ∈ [0,1]. For each t ∈ [0, ∞) and x, y ∈ X, define (M, N) by

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + |x - y|}, & t > 0, \\ 0, & t = 0 \end{cases}$$

and $$N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0, \\ 1, & t = 0 \end{cases}.$$

Clearly, (X, M, N, *, ◊) is an intuitionistic fuzzy metric space.

Define $g(x) = \dfrac{x}{6}$ and $f(x) = \dfrac{x}{2}$ on X. It is clear that $g(x) \subseteq f(x)$.

Now, $$M\left(gx, gy, \frac{t}{3}\right) = \frac{\frac{t}{2}}{\frac{t}{2} + \frac{|x-y|}{6}} = \frac{3t}{3t+|x-y|} \geq \frac{t}{t + \frac{|x-y|}{3}} = \frac{3t}{3t+|x-y|} = M(fx, fy, t),$$

and

$$N\left(gx, gy, \frac{t}{3}\right) = \frac{\frac{|x-y|}{6}}{\frac{t}{2} + \frac{|x-y|}{6}} = \frac{|x-y|}{3t+|x-y|} \leq \frac{\frac{|x-y|}{3}}{t + \frac{|x-y|}{3}} = \frac{|x-y|}{3t+|x-y|} = N(fx, fy, t).$$

Thus, all the conditions of Theorem 3.1 are satisfied and so f and g have a unique common fixed point 0.

As an application of Theorem 3.1, we prove a common fixed point theorem for four finite families of mappings which runs as follows:

Theorem 3.2. Let $\{f_1, f_2, \ldots, f_m\}$ and $\{g_1, g_2, \ldots, g_n\}$ be two finite families of self-mappings of a intuitionistic fuzzy metric spaces with continuous t-norm * and continuous t-conorm ◊ defined by $a*a \geq a$ and $(1-a) \lozenge (1-a) \leq (1-a)$ for all $a \in [0, 1]$ such that $f = f_1 f_2 \ldots f_m$, $g = g_1 g_2 \ldots g_n$, satisfy condition (3.1), (3.2) and (3.3).

Then f and g have a point of coincidence. Moreover, if $f_i f_j = f_j f_i$, $g_k g_l = g_l g_k$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, n\}$, then (for all $i \in I_1$, $k \in I_2$) f_i and g_k have a common fixed point.

Proof. The conclusions is immediate i.e., f and g have a point of coincidence as f and g satisfy all the conditions of Theorem 3.1. Now appealing to component wise commutativity of various pairs, one can immediately prove that fg = gf, hence, obviously pair (f, g) is occasionally weakly compatible. Note that all the conditions of Theorem 3.1 are satisfied which ensured the existence of a unique common fixed point, say z. Now one need to show that z remains the fixed point of all the component maps.

For this consider

$$\begin{aligned} f(f_i z) &= ((f_1 f_2 \ldots f_m) f_i) z = (f_1 f_2 \ldots f_{m-1})((f_m f_i) z) = (f_1 \ldots f_{m-1})(f_i f_m z) \\ &= (f_1 \ldots f_{m-2})(f_{m-1} f_i (f_m z)) = (f_1 \ldots f_{m-2})(f_i f_{m-1}(f_m z)) \\ &= \ldots f_1 f_i (f_2 f_3 f_4 \ldots f_m z) = f_i f_1 (f_2 f_3 \ldots f_m z) \\ &= f_i(fz) = f_i z. \end{aligned}$$

Similarly, one can show that

$$f(g_k z) = g_k(fz) = g_k z,\ g(g_k z) = g_k(gz) = g_k z$$

and $$g(f_i z) = f_i(gz) = f_i z,$$

which show that (for all i and k) $f_i z$ and $g_k z$ are other fixed points of the pair (f, g).

Now appealing to the uniqueness of common fixed points of both pairs separately, we get

$z = f_i z = g_k z$, which shows that z is a common fixed point of f_i, g_k for all i and k.

IV. CONCLUSION

Theorem 3.1 is a generalization of the result of Turkoglu et. al. [9] in the sense that condition of commuting mappings of the pairs of self maps has been restricted to occasionally weakly compatible self maps.

REFERENCES

[1] Alaca, C., Turkoglu, D. and Yildiz, C., Fixed points in Intuitionistic fuzzy metric spaces, Smallerit Chaos, Solitons & Fractals, 29 (5) (2006), 1073 – 1078.
[2] Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and System, 20 (1986), 87 – 96.
[3] Coker, D., An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and System, 88(1997), 81–89.
[4] George, A. and Veeramani, P., On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64 (1994), 395 – 399.
[5] Jungck, G., Commuting mappings and fixed points, Amer. Math. Monthly, 83(1976), 261–263.
[6] Kramosil, O. and Michalek, J., Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 336-344.
[7] Pant, R.P., Common fixed points of non-commuting mappings, J. Math. Anal. Appl., 188(1994), 436–440.
[8] Park, J.H., Intuitionistic fuzzy metric spaces, Chaos, Solitions & Fractals, 22 (2004), 1039-1046.
[9] Turkoglu, D., Alaca, C., Cho, Y.J., and Yildiz, C., Common fixed point theorems in intuitionistic fuzzy metric space, J. Appl. Math. and Computing, 22 (2006), 411 - 424.
[10] Zadeh, L. A., Fuzzy sets, Inform and control 89 (1965), 338-353.