FIXED POINT THEOREMS IN INTUITIONISTIC FUZZY METRIC SPACE

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ABSTRACT
In this paper, common fixed point theorems for occasionally weakly compatible mappings in intuitionistic fuzzy metric space has been proved which is a generalization of the result of Turkoglu et. al. [9]. We also cited an example in support of our result.

Keywords - Common Fixed Points, Intuitionistic Fuzzy Metric Space, Compatible Maps And Occasionally Weakly Compatible Mappings

AMS Subject Classification: Primary 47H10, Secondary 54H25.

I. INTRODUCTION
Zadeh’s [10] investigation of the notion of fuzzy set has led to a rich growth of fuzzy Mathematics. Many authors have introduced the concept of fuzzy metric in different ways. Atanassov [2] introduced and studied the concept of intuitionistic fuzzy set. The notion of intuitionistic fuzzy metric space defined by Park [8] is a generalization of fuzzy metric space due to George and Veeramani [4]. Further, using the idea of Intuitionistic Fuzzy set, Alaca et.al. [1] defined the notion of Intuitionistic Fuzzy Metric space, as Park [8] with the help of continuous t – norms and continuous t–conorms, as a generalization of fuzzy metric space due to Kramosil and Michalek [6], further Coker [3], Turkoglu [9] and others have been expansively developed the theory of Intuitionistic Fuzzy set and applications. Turkoglu et. al. [9] introduced the notion of Cauchy sequences in intuitionistic fuzzy metric space. They generalized the Jungck’s [5] common fixed point theorem in intuitionistic fuzzy metric space and proved the intuitionistic fuzzy version of Pant’s theorem [7] by giving the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric space.

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.
II. PRELIMINARIES

**Definition 2.1.** [1] A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t-norm if $*$ satisfies the following conditions:

- (i) $*$ is commutative and associative;
- (ii) $*$ is continuous;
- (iii) $a * 1 = a$ for all $a \in [0, 1]$;
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

**Definition 2.2.** [1] A binary operation $\triangledown: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t-conorm if $\triangledown$ satisfies the following conditions:

- (i) $\triangledown$ is commutative and associative;
- (ii) $\triangledown$ is continuous;
- (iii) $a \triangledown 0 = a$ for all $a \in [0, 1]$;
- (iv) $a \triangledown b \leq c \triangledown d$ whenever $a \leq c$ and $b \leq d$
  for all $a, b, c, d \in [0, 1]$.

**Remark 2.1.** [1] The concepts of triangular norms (t-norms) and triangular co-norms (t-conorms) are known as axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively.

**Definition 2.3.** [1] A 5-tuple $(X, M, N, *, \triangledown)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm, $\triangledown$ is a continuous t-conorm and $M, N$ are fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

- (i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
- (ii) $M(x, y, 0) = 0$ for all $x, y \in X$;
- (iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
- (iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
- (v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$;
- (vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is left continuous;
- (vii) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
- (viii) $N(x, y, 0) = 1$ for all $x, y \in X$;
- (ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
- (x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
N(x, y, t) \geq N(y, z, s) \geq N(x, z, t + s) for all x, y, z \in X and s, t > 0;

for all x, y \in X, N(x, y, .) : [0, \infty) \to [0,1] is right continuous;

\lim_{t \to x} N(x, y, t) = 0 for all x, y \in X.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions M(x,y,t) and N(x,y,t) denote the degree of nearness and the degree of non nearness between x and y with respect to t, respectively.

**Remark 2.1.** [1] Every fuzzy metric space \((X, M, *)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1-M, *, \emptyset)\) such that t-norm * and t-conorm are associated, i.e.,

\[ x \bowtie y = 1 - (1 - x) * (1 - y) \quad \text{for all } x, y \in X. \]

**Remark 2.2.** [1] In intuitionistic fuzzy metric space \(X, M(x, y, .)\) is non-decreasing and \(N(x, y, .)\) is non-increasing for all \(x, y \in X\).

**Definition 2.4.** [1] Let \((X, M, N, *, \emptyset)\) be an intuitionistic fuzzy metric space. Then

(a) a sequence \(\{x_n\}\) in X is said to be Cauchy sequence if, for all \(t > 0\) and \(p > 0\),

\[ \lim_{s \to \infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{s \to \infty} N(x_{n+p}, x_n, t) = 0. \]

(b) a sequence \(\{x_n\}\) in X is said to be convergent to a point \(x \in X\) if, for all \(t > 0\),

\[ \lim_{s \to \infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{s \to \infty} N(x_n, x, t) = 0. \]

**Definition 2.5.** [1] An intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) is said to be complete if and only if every Cauchy sequence in X is convergent.

**Lemma 2.1.** [2] Let \((X, M, N, *, \emptyset)\) be an intuitionistic fuzzy metric space. If there exists \(k \in (0,1)\) such that

\[ M(x,y,kt) \geq M(x,y,t) \quad \text{and} \quad N(x,y,kt) \leq N(x,y,t) \quad \text{for } x, y \in X. \]

Then \(x = y\).

**Lemma 2.2.** [2] Let \((X, M, N, *, \emptyset)\) be an intuitionistic fuzzy metric space. If there exists a number \(k \in (0,1)\) such that

(a) \(M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \quad N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)\)

for all \(t > 0\) and \(n = 1, 2, ...\) then \(\{y_n\}\) is a Cauchy sequence in X.

**Definition 2.6.** [1] Two maps A and B from an intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) into itself are said to be compatible if

\[ \lim_{k \to \infty} M(ABx_n, BAX_n, t) = 1 \quad \text{and} \quad \lim_{k \to \infty} N(ABx_n, BAX_n, t) = 0 \]

for all \(t > 0\), whenever \(\{x_n\}\) is a sequence in X such that
\[ \lim_{n \to \infty} A x_n = \lim_{n \to \infty} B x_n = x \quad \text{for some } x \in X. \]

**Definition 2.7.** Two maps \( A \) and \( B \) from an intuitionistic fuzzy metric space \((X, M, N, *, \Diamond)\) into itself are said to be **occasionally weakly compatible** if and only if there is a point \( x \) in \( X \) which is coincidence point of \( A \) and \( B \) at which \( A \) and \( B \) commute.

### III. MAIN RESULT

In this section, we prove a common fixed point theorem for occasionally weakly compatible mappings in intuitionistic fuzzy metric space which is a generalization of Turkoglu et al. [9].

**Theorem 3.1.** Let \((X, M, N, *, \Diamond)\) be an intuitionistic fuzzy metric spaces. Let \( f, g : X \to X \) be occasionally weakly compatible mappings satisfying the following conditions:

1. \((3.1)\) \( g(X) \subseteq f(X); \)
2. \((3.2)\) there exists a number \( k \in (0, 1) \) such that
   
   \[ M(gx, gy, kt) \geq M(fx, fy, t) \]
   
   \[ N(gx, gy, kt) \leq N(fx, fy, t) \]
   
   for all \( x, y \in X \) and \( t > 0 \);
3. \((3.3)\) one of the subspaces \( g(X) \) or \( f(X) \) is complete.

Then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** By (3.1), since \( g(X) \subseteq f(X) \), for any \( x_0 \in X \), there exists a point \( x_1 \in X \) such that \( gx_0 = fx_1 \).

In general, chose \( x_{n+1} \) such that

\[ y_n = fx_{n+1} = gx_n \]

From Turkoglu et al. [9], we conclude that \( \{ y_n \} \) is a Cauchy sequence in \( X \).

Since either \( f(X) \) or \( g(X) \) is complete, for definiteness assume that \( f(X) \) is complete.

Since \( f(X) \) is complete, so there exists a point \( p \in X \) such that

\[ fp = z. \]

Now using (3.2), we have

\[ M(gp, gx_n, kt) \geq M(fp, fx_n, t) \]

and

\[ N(gp, gx_n, kt) \leq N(fp, fx_n, t). \]

Taking limit as \( n \to \infty \), we obtain

\[ gp = z. \]

Therefore, we have \( fp = gp = z \).
Since \( f \) and \( g \) are occasionally weakly compatible, therefore
\[
fgp = gfp, \text{ i.e., } fz = gz.
\]

Now we show that \( z \) is a common fixed point of \( f \) and \( g \).

From (3.2), we get
\[
M(gz, gx_n, kt) \geq M(fz, fx_n, t)
\]
and
\[
N(gz, gx_n, kt) \leq N(fz, fx_n, t)
\]

Proceeding limit as \( n \rightarrow \infty \), we obtain
\[
gz = z.
\]

Hence \( z \) is a common fixed point of \( f \) and \( g \) both.

**Uniqueness**: Let \( w \) be another common fixed point of \( f \) and \( g \) then
\[
fw = gw = w.
\]

Using (3.2), we get
\[
M(gz, gw, kt) \geq M(fz, fw, t)
\]
\[
N(gz, gw, kt) \leq N(fz, fw, t)
\]
or
\[
M(z, w, kt) \geq M(z, w, t)
\]
\[
N(z, w, kt) \leq N(z, w, t)
\]

Using lemma 2.1, we get
\[
z = w.
\]

Hence, \( z \) is the unique common fixed point of \( f \) and \( g \).

This completes the proof.

**Example 3.1.** Let \( X = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{ 0 \} \) with \( * \) continuous t-norm and \( \oplus \) continuous t-conorm defined by \( a * b = ab \) and \( a \oplus b = \min \{ 1, a+b \} \) respectively, for all \( a, b \in [0,1] \). For each \( t \in [0, \infty) \) and \( x, y \in X \), define \( (M, N) \) by
\[
M(x, y, t) = \begin{cases} 
\frac{t}{t + |x - y|}, & t > 0, \\
0, & t = 0 
\end{cases}
\]
and \[ N(x, y, t) = \begin{cases} \frac{|x - y|}{t + |x - y|}, & t > 0, \\ 1, & t = 0. \end{cases} \]

Clearly, \((X, M, N, *, \varnothing)\) is an intuitionistic fuzzy metric space.

Define \(g(x) = \frac{x}{6}\) and \(f(x) = \frac{x}{2}\) on \(X\). It is clear that \(g(x) \subseteq f(x)\).

Now, \(M\left(\frac{g(x, y)}{3}\right) = \frac{2}{t + \frac{|x - y|}{6}} = \frac{3t}{t + \frac{|x - y|}{3}} = \frac{3t}{3 + \frac{|x - y|}{3}} = M\left(\frac{f(x, y)}{t}\right)\),

and

\[ N\left(\frac{g(x, y)}{3}\right) = \frac{|x - y|}{3 + \frac{|x - y|}{3}} \leq \frac{|x - y|}{3 + \frac{|x - y|}{3}} = N\left(\frac{f(x, y)}{t}\right). \]

Thus, all the conditions of Theorem 3.1 are satisfied and so \(f\) and \(g\) have a unique common fixed point 0.

As an application of Theorem 3.1, we prove a common fixed point theorem for four finite families of mappings which runs as follows:

**Theorem 3.2.** Let \(\{f_1, f_2, \ldots, f_m\}\) and \(\{g_1, g_2, \ldots, g_n\}\) be two finite families of self-mappings of an intuitionistic fuzzy metric spaces with continuous \(t\)-norm * and continuous \(t\)-conorm \(\varnothing\) defined by \(a* a \geq a\) and \((1-a) \varnothing (1-a) \leq (1-a)\) for all \(a \in [0, 1]\) such that \(f = f_1 f_2 \ldots f_m\), \(g = g_1 g_2 \ldots g_n\), satisfy condition (3.1), (3.2) and (3.3).

Then \(f\) and \(g\) have a point of coincidence. Moreover, if \(f_i f_j = f_j f_i\) and \(g_k g_l = g_l g_k\) for all \(i, j \in I_1 = \{1, 2, \ldots, m\}\), \(k, l \in I_2 = \{1, 2, \ldots, n\}\), then (for all \(i \in I_1, k \in I_2\)) \(f_i\) and \(g_k\) have a common fixed point.

**Proof.** The conclusion is immediate i.e., \(f\) and \(g\) have a point of coincidence as \(f\) and \(g\) satisfy all the conditions of Theorem 3.1. Now appealing to component wise commutativity of various pairs, one can immediately prove that \(fg = gf\), hence, obviously pair \((f, g)\) is occasionally weakly compatible. Note that all the conditions of Theorem 3.1 are satisfied which ensured the existence of a unique common fixed point, say \(z\). Now one need to show that \(z\) remains the fixed point of all the component maps.

For this consider
\[ f(fz) = ((f_1 f_2 \ldots f_m)z) = (f_1 f_2 \ldots f_{m-1})(f_m f_i)z = (f_1 \ldots f_{m-1})(f_m f_i z) = (f_1 \ldots f_{m-2})(f_{m-1} f_i f_m z) = \ldots = f_i f_j f_3 \ldots f_m z = f_i f_j f_3 \ldots f_m z = f_i (f_j z) = f_i z. \]
Similarly, one can show that

\[ f(g_kz) = g_k(fz) = g_kz, \quad g(g_kz) = g_k(gz) = g_kz \]

and

\[ g(f_i z) = f_i(gz) = f_i z, \]

which show that (for all \( i \) and \( k \)) \( f_i z \) and \( g_k z \) are other fixed points of the pair \((f, g)\).

Now appealing to the uniqueness of common fixed points of both pairs separately, we get

\[ z = f_i z = g_k z, \]

which shows that \( z \) is a common fixed point of \( f_i, g_k \) for all \( i \) and \( k \).

IV. CONCLUSION

Theorem 3.1 is a generalization of the result of Turkoglu et. al. [9] in the sense that condition of commuting mappings of the pairs of self maps has been restricted to occasionally weakly compatible self maps.

REFERENCES


